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A UNIQUENESS THEOREM FOR TRIGONOMETRIC SERIES

By S. VERBLUNSKY

[Received 24 November 1930]

1. THE object of this note is to prove the following theorem:

THEOREM A. *Let the series*

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \qquad a_n = o(n),\ b_n = o(n), \quad (1)$$

have its upper and lower Poisson sums $\overline{P}(x)$, $\underline{P}(x)$ *finite at every point, and let*

$$\underline{P}(x) \geqslant \psi(x) \quad (2)$$

where $\psi(x)$ *is integrable in the sense of Denjoy-Perron in* $(0, 2\pi)$. *Then* (1) *is a Fourier-Denjoy series.*

If $$P(r, x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n,$$

then the result remains true when the Poisson sums are not finite at the points of an enumerable set E, *provided that for* $x \subset E$, *the condition*

$$\lim_{r \to 1} (1-r) P(r, x) = 0 \quad (3)$$

is satisfied.

This theorem is related to some investigations given in a previous paper,* a knowledge of which will be assumed. The object of that paper was to obtain the analogues of certain uniqueness theorems due to Zygmund,† when Zygmund's condition (F), namely, that

$$\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}$$

be the Fourier series of a continuous function, is replaced by the conditions

$$a_n = o(n), \qquad b_n = o(n).$$

In that paper the analogue of one theorem of Zygmund was not obtained. This omission is filled by the present note.

The enunciation of Theorem A renders it necessary to observe that

* Verblunsky, *Proc. London Math. Soc.* (2), 31 (1930), 370–86. Referred to as I.

† Zygmund, *Math. Zeit.* 25 (1926), 274–90.

the method of proof shows that if $\psi(x)$ is integrable L, then (1) is a Fourier L series.

We shall suppose in all that follows that $a_0 = 0$.

2. In this section we are concerned with three preliminary lemmas.

LEMMA 1. *If $F(x)$ is integrable L, and*

$$F(x) \sim -\sum \frac{a_n \cos nx + b_n \sin nx}{n^2} \tag{4}$$

is at $x = \xi$ summable by Poisson's method, then

$$\overline{D}^2 F(\xi) \geqslant \underline{P}(\xi), \qquad \underline{D}^2 F(\xi) \leqslant \overline{P}(\xi).$$

This lemma is due to Rajchman and Zygmund.* The enunciation is more general than theirs, but this is the result which is established by their proof.

LEMMA 2. *If $F(x)$ be upper semi-continuous in an interval (α, β) containing the interval (a, b) in its interior, and if $\overline{D}^2 F(x) \geqslant 0$ at every interior point of (α, β), then*

$$F(x) \leqslant F(a) + \frac{x-a}{b-a}\{F(b)-F(a)\} \qquad (a < x < b). \tag{5}$$

If not, there is a point c such that

$$F(c) > F(a) + \frac{c-a}{b-a}\{F(b)-F(a)\} \qquad (a < c < b).$$

Writing
$$\Phi(x) = F(x) + \epsilon(x-a)^2 \qquad (\epsilon > 0),$$
we can choose ϵ such that

$$\Phi(c) > \Phi(a) + \frac{c-a}{b-a}\{\Phi(b)-\Phi(a)\}.$$

Consider the function

$$\phi(x) = \Phi(x) - \Phi(a) - \frac{x-a}{b-a}\{\Phi(b)-\Phi(a)\}.$$

We have
$$\phi(a) = \phi(b) = 0, \qquad \phi(c) > 0.$$
Further, $\phi(x)$ is upper semi-continuous. Hence it attains its maximum at a point ξ satisfying $a < \xi < b$. At ξ,

$$\overline{D}^2 \Phi(\xi) = \overline{D}^2 \phi(\xi) \leqslant 0.$$

But
$$\overline{D}^2 \Phi(\xi) = \overline{D}^2 F(\xi) + 2\epsilon.$$
Hence
$$\overline{D}^2 F(\xi) \leqslant -2\epsilon,$$
which contradicts the hypothesis.

* Rajchman and Zygmund, *Math. Zeit.* 25 (1926), 261–73. Cf. in particular the footnote to their Lemma II.

LEMMA 3. *If $F(x)$ be upper semi-continuous in an interval (α, β), and if $\bar{D}^2F(x) \geqslant 0$ at each point of the interval, then $F(x)$ is continuous in the open interval (α, β).*

From the relation (5) we infer that, for $\alpha < x < y < z < \beta$,

$$\frac{F(x)-F(y)}{x-y} \leqslant \frac{F(x)-F(z)}{x-z}$$

or, what is the same,

$$\frac{F(x)-F(y)}{x-y} \leqslant \frac{F(y)-F(z)}{y-z}.$$

Hence, for $\alpha < x < y < z < w < \beta$, we have

$$\frac{F(x)-F(y)}{x-y} \leqslant \frac{F(z)-F(w)}{z-w}.$$

Let x tend to y and z to w in a suitable manner and we infer that

$$\bar{D}F(y) \leqslant \bar{D}F(w).$$

In other words, $\bar{D}F(x)$ is a monotone-increasing function in the open interval (α, β). A similar argument shows that each of the other extreme derivatives is monotone-increasing. Hence, in any interval strictly interior to (α, β), all four derivatives are bounded. This proves the lemma.

3. We turn now to the proof of Theorem A, and consider the case in which the Poisson sums are everywhere finite. We may write

$$F(x) = -\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}, \tag{6}$$

the series being everywhere convergent. Then $F(x)$ possesses the property R (I, Lemma 4): i.e. the points of discontinuity of $F(x)$, with respect to any perfect set P, are non-dense on P. Further, by Lemma 1,

$$\bar{D}^2F(x) \geqslant \underline{P}(x) \geqslant \psi(x) \tag{7}$$

at every point.

We consider an arbitrarily assigned interval $\Delta = (a, b)$, and supposing, as we may, that $\psi(x)$ is everywhere finite, we denote by $\phi(x)$ a minor function of $\psi(x)$ in (a, b). Writing

$$\Phi(x) = \int_a^x \phi(t)\, dt$$

and

$$G(x) = F(x) - \Phi(x),$$

we have

$$\bar{D}^2G(x) \geqslant 0 \tag{8}$$

at every interior point of Δ. Further, $G(x)$ possesses the property R within (a, b), and is, like $F(x)$, a differential coefficient.

We now show that $G(x)$ cannot have an isolated discontinuity within Δ. For suppose that ξ is such a discontinuity. Then there is an $\epsilon > 0$ such that ξ is the only point of discontinuity of $G(x)$ in $(\xi-\epsilon, \xi+\epsilon)$. Since $G(x)$ is continuous in $(\xi-\epsilon \leqslant x < \xi)$, it follows by the proof of Lemma 3 that $G(x)$ has increasing derivatives in that interval, and hence, that there is a constant K such that

$$G(x)+Kx$$

is monotone-increasing in $(\xi-\frac{1}{2}\epsilon \leqslant x < \xi)$.

In a similar manner, we can find a constant L such that

$$G(x)+Lx$$

is monotone-diminishing in $(\xi < x \leqslant \xi+\frac{1}{2}\epsilon)$.

The function $H(x)$ defined by

$$H(x) = G(x)+Kx \qquad (\xi-\tfrac{1}{2}\epsilon \leqslant x < \xi)$$

$$H(\xi) = G(\xi)+K\xi$$

$$H(x) = G(x)+Lx+(K-L)\xi \qquad (\xi < x \leqslant \xi+\tfrac{1}{2}\epsilon)$$

is continuous in $(\xi-\frac{1}{2}\epsilon \leqslant x \leqslant \xi+\frac{1}{2}\epsilon)$. From the definition of this function it follows that $G(x)$ can have only an ordinary discontinuity at ξ, and since $G(x)$ is a differential coefficient this is impossible.

4. We have thus proved that the points of discontinuity of $G(x)$ on Δ form a set E which is dense in itself; and this set E is everywhere dense on a perfect set P.

We can in virtue of the property R find an interval $d \subset \Delta$ the end points of which are points of P, such that $G(x)$ is continuous on Pd, while its points of discontinuity on d are everywhere dense in Pd.

Let $(\alpha_n, \beta_n) \subset d$ be an interval contiguous to Pd. In the open interval (α_n, β_n) $G(x)$ is continuous, and by (8) and the proof of Lemma 3, has increasing derivatives. It follows that $G(x)$ cannot have a discontinuity of the second kind at α_n on the right, nor at β_n on the left. As a differential coefficient it cannot have an ordinary discontinuity, so that $G(x)$ is continuous in the closed interval (α_n, β_n). Furthermore, from the fact that $G(x)$ has increasing derivatives within (α_n, β_n), we infer that $G(x)$ must attain its upper bound in the closed interval (α_n, β_n) at an end point. Since, then, $G(x)$ is continuous on Pd, it must be upper semi-continuous on d. By (8) and

Lemma 3, it must be continuous on d. This contradicts the assumption that E exists. Hence G is continuous in Δ.

In virtue of (8) and the proof of Lemma 3,

$$G(x) = F(x) - \int_a^x \phi(t)\, dt$$

has increasing derivatives within Δ. If we choose a sequence of minor functions $\phi_n(x)$ which tend uniformly to

$$\int_a^x \psi(t)\, dt,$$

we infer that
$$F(x) - \int_a^x dy \int_a^y \psi(t)\, dt \tag{9}$$

has increasing derivatives within Δ. It follows that the expression (9) has almost everywhere in Δ a second differential coefficient which is integrable L in $(a+\epsilon, b-\epsilon)$. But

$$\int_a^x dy \int_a^y \psi(t)\, dt \tag{10}$$

has almost everywhere in Δ a second differential coefficient $\psi(x)$ which is integrable D in Δ. Hence $F(x)$ has almost everywhere in Δ a second differential coefficient which is integrable D in $(a+\epsilon, b-\epsilon)$. Since Δ may be arbitrarily chosen, it follows that $D^2F(x)$ exists almost everywhere and is integrable D in every finite interval. At a point ξ at which $D^2F(x)$ does not exist, there is, by Lemma 1, since $\overline{P}(\xi)$, $\underline{P}(\xi)$ are both finite, a finite number between $\overline{D}^2F(\xi)$ and $\underline{D}^2F(\xi)$. Hence, by a classical theorem of de la Vallée Poussin,* (1) is a Fourier-Denjoy series.

5. We consider now the case in which the Poisson sums are not both finite at the points of an enumerable set E, at which the condition (3) is satisfied. This requires a somewhat difficult argument. We begin by stating the following lemma which is implied by the work of Zygmund (loc. cit., Theorem III).

LEMMA 4. *Under the conditions of Theorem A, if* (α, β) *be interior to an interval in which*

$$-\sum \frac{a_n \cos nx + b_n \sin nx}{n^2}$$

converges to a continuous function $F(x)$, *then* $D^2F(x)$ *exists almost everywhere in* (α, β) *and is integrable* D *in this interval.*

* de la Vallée Poussin, *Bull. de l'Acad. Roy. de Belg.* (1912), pp. 701–7.

We next prove

LEMMA 5. *If at the points of an interval (a, b) containing the interval (α, β) in its interior, the upper and lower Poisson sums of (1) are finite except at the points of an enumerable set E; if, for x not belonging to E,*

$$\underline{P}(x) \geqslant \psi(x)$$

where $\psi(x)$ is integrable D in (a, b), while for $x \subset E$ condition (3) is satisfied; if further the series

$$F(x) = -\sum \frac{a_n \cos nx + b_n \sin nx}{n^2}$$

converges everywhere in (a, b), then $F(x)$ is continuous in (α, β). And, if $\phi(x)$ denote a minor function of $\psi(x)$ in (a, b) and

$$\Phi(x) = \int_a^x \phi(t)\, dt,$$

then

$$\bar{D}^2\{F(x) - \Phi(x)\} \geqslant 0$$

at all points of (α, β).

The set E of points at which the Poisson sums are not both finite is an ordinary inner limiting set.* Being enumerable, it contains no component which is dense in itself. We can write

$$E = E^1 = E_1 + E^2$$

where E_1 is the set of isolated points of E^1 and is not null. Similarly, we can write, if E^2 is not null,

$$E^2 = E_2 + E^3$$

where E_2 is the set of isolated points of E^2 and is not null. Proceeding in this manner, we have since E is enumerable,

$$E = \sum_{\beta < \gamma} E_\beta$$

where γ is some ordinal of the first or second class.

Consider a point $x_0 \subset E_1$. There is an $\epsilon > 0$ such that $(x_0 - 2\epsilon, x_0 + 2\epsilon)$ contains no point of E other than x_0. For all x in $\delta \equiv (x_0 - \epsilon, x_0 + \epsilon)$ we have, by Lemma 1,

$$\bar{D}^2 F(x) \geqslant \psi(x).$$

Hence $$\bar{D}^2\{F(x) - \Phi(x)\} \geqslant 0 \qquad (x \neq x_0, x < \delta). \tag{11}$$

* Cf. Zygmund, loc. cit., footnote, p. 285.

As in §§ 3, 4, we can show that

$$G(x) = F(x) - \Phi(x)$$

is continuous in $(x_0 - \epsilon \leqslant x < x_0)$ and in $(x_0 < x \leqslant x_0 + \epsilon)$. Further, by the argument at the end of § 3, $G(x)$ is either continuous at x_0 or has an ordinary discontinuity at x_0. The latter is impossible. Thus $F(x)$ is continuous in δ.

We now apply Lemma 4 to show that (11) is satisfied at x_0. By that lemma, $D^2F(x) = g(x)$ exists almost everywhere in δ and is integrable D. In $(x_0 - \epsilon \leqslant x < x_0)$ we have

$$F(x) = \int_{x_0-\epsilon}^{x} dy \int_{x_0-\epsilon}^{y} g(t)\,dt + Ax + B, \tag{12}$$

and in $(x_0 < x \leqslant x_0 + \epsilon)$ we have

$$F(x) = \int_{x}^{x_0+\epsilon} dy \int_{y}^{x_0+\epsilon} g(t)\,dt + A'x + B'.$$

Hence
$$\lim_{h\to 0} \frac{F(x_0+h)+F(x_0-h)-2F(x_0)}{h} = l(x_0)$$

exists. It can easily be shown that the existence of $l(x_0)$ implies that

$$\lim_{r\to 1} (1-r)P(r, x_0) = l(x_0).$$

By (3) we have $l(x_0) = 0$. It now follows that (12) holds for all x in δ. Hence

$$\lim_{h\to 0} \frac{F(x+h)+F(x-h)-2F(x)}{h} = 0$$

for all x in δ. Clearly

$$\lim_{h\to 0} \frac{\Phi(x+h)+\Phi(x-h)-2\Phi(x)}{h} = 0$$

for all x in δ, so that $G(x)$ has the same property. If, then, (11) did not hold at x_0, it would not hold at an unenumerable set of points in δ, a conclusion we know to be false.

It thus follows that $F(x)$ is continuous in any interval which contains no point of E^2, and that (11) holds at every point of that interval. We consider next an isolated point of E^2, say $x_0 \subset E_2$. The above argument can be repeated precisely. By transfinite induction we thus establish the lemma.

6. We now show that under the conditions of Theorem A, the

series obtained by two formal integrations of (1) converges everywhere. We first observe that the continuous function

$$\mathfrak{F}(x) = -\sum \frac{a_n \sin nx - b_n \cos nx}{n^3} \tag{13}$$

has a differential coefficient equal to

$$-\sum \frac{a_n \cos nx + b_n \sin nx}{n^2} \tag{14}$$

for x not belonging to E. We can show that for $\xi \subset E$, the four derivatives of $\mathfrak{F}(x)$ are bounded. For in virtue of (3) we have

$$\sum \frac{a_n \cos n\xi + b_n \sin n\xi}{n} r^n = o\left(\log \frac{1}{1-r}\right)$$

and

$$\sum \frac{a_n \cos n\xi + b_n \sin n\xi}{n^2} r^n = O(1).$$

Since the coefficients in the last series are $o(1/n)$, the partial sums of

$$\sum \frac{a_n \cos n\xi + b_n \sin n\xi}{n^2}$$

are bounded. It follows* that the limits of

$$\frac{\mathfrak{F}(\xi+h) - \mathfrak{F}(\xi-h)}{2h}$$

as $h \to 0$ are bounded. On the other hand

$$\lim_{h \to 0} \frac{\mathfrak{F}(\xi+h) + \mathfrak{F}(\xi-h) - 2\mathfrak{F}(\xi)}{h} = 0$$

since the coefficients in (13) are $o(1/n^2)$. This establishes our assertion.

We denote the sum of (14) at the points at which it converges by $F(x)$. We proceed to show that it converges everywhere. The set of points of non-convergence are contained in E and therefore contain no component which is dense in itself. This set therefore has an isolated point ξ. We can find an $\epsilon > 0$ such that ξ is the only point of non-convergence in $(\xi - 2\epsilon, \xi + 2\epsilon)$. Let $\delta \equiv (\xi - \epsilon, \xi + \epsilon)$, and let $\phi(x)$ be a minor function of $\psi(x)$ in δ. Then by Lemma 5, $F(x)$ is continuous in $(\xi - \epsilon \leqslant x < \xi)$ and in $(\xi < x \leqslant \xi + \epsilon)$, and

$$\overline{D}^2\{F(x) - \Phi(x)\} \geqslant 0 \qquad (x \neq \xi,\ x \subset \delta).$$

Writing

$$G(x) = F(x) - \Phi(x),$$

it follows, as in § 3, that there is a constant K such that

$$G(x) + Kx$$

* Cf. the argument on p. 636 of Hobson, *Theory of Functions of a Real Variable*, vol. ii.

is monotone-increasing in $(\xi-\epsilon \leqslant x < \xi)$. Hence $G(\xi-0)$ has a meaning, and is finite or infinite. The same applies to $F(\xi-0)$ and to $F(\xi+0)$. Since the derivatives of $\mathfrak{F}(x)$ at ξ are bounded, $F(\xi-0)$ and $F(\xi+0)$ are both finite. Hence (14) is summable by Poisson's method at $x=\xi$, and so, by Tauber's theorem, is convergent at that point.

Since (14) converges everywhere, denoting its sum by $F(x)$, it follows by Lemma 5 that $F(x)$ is continuous everywhere, and so by Lemma 4 that $D^2F(x)$ exists almost everywhere and is integrable D in every finite interval. Writing $D^2F(x)=f(x)$ wherever the first member exists, we have, in any interval free from points of E,

$$F(x)=\int_a^x dy \int_a^y f(t)\,dt+Ax+B \qquad (a<x<b).$$

By the argument on pp. 382–3 of I, we can show that this relation holds in every finite interval, and this establishes the theorem.

A PROBLEM CONCERNING THE HYPERGEOMETRIC EQUATION (II*)

By E. G. C. POOLE

[Received 13 January 1931]

7. Resuming the investigation of cases where the hypergeometric equation has n distinct solutions, whose product is monodromic, we consider in this paper the values $n = 4$ and $n = 5$. It appears that, for all values of n, there is a solution of the problem in which the equation has an apparent singularity and admits two independent monodromic integrals, and another solution when the equation has two quadratic singularities and is reducible to the form considered by Elliot and Darboux. Apart from these solutions, which are explained in (8) and (9), and which were already prominent in the previous paper, there is no solution for $n = 2$, 3, or 5.

For $n = 4$, there is a different kind of solution, when the parameters of the hypergeometric equation differ by integers from those of certain reduced forms. Two singularities at least must be of the third order, the remaining one being of either the second or the third order. But this condition, though necessary, is not sufficient. By elementary processes, taking the points $x = 0$ and $x = 1$ as points of the third order, and assigning the exponent zero to the appropriate solution at each of these points, the reduced forms, in which we take $1 > a \geqslant b \geqslant 0$, $1 > c \geqslant 0$, are found to be

$$F(\tfrac{3}{4}, \tfrac{1}{4}; \tfrac{1}{3}; x), \qquad F(\tfrac{3}{4}, \tfrac{1}{4}; \tfrac{2}{3}; x), \qquad F(\tfrac{1}{2}, \tfrac{1}{6}; \tfrac{1}{3}; x), \qquad F(\tfrac{5}{6}, \tfrac{1}{2}; \tfrac{2}{3}; x).$$

The examination of this solution occupies the greater part of (10); the remaining paragraphs of (10) and the whole of (11) are occupied in reviewing and rejecting all other possible types of solution.

8. It was seen in (3.2) and (5.4) that, when one singularity is apparent, the other two are of the same order, and the type of the equation is $(N, N, 1)$; excluding equations with logarithmic points, which cannot yield a solution of our problem, there are two independent monodromic solutions, which may be taken in the canonical form

$$y_1 = F(a, b; c; x), \qquad y_2 = x^{1-c}(1-x)^{c-a-b}F(1-a, 1-b; 2-c; x),$$

* *Quart. J. of Math.* (Oxford), 1 (1930), 108–15. We take the opportunity of noting the following corrections: p. 111, line 4, *for* (2.6) *read* (2.16); p. 114, line 3, *for* $q^3 = 1$ *read* $s^3 = 1$; p. 114, line 5, *for* $q = 1$ *read* $s = 1$.

where a, b are integers, $a \leqslant 0 < 1 \leqslant b$, and c is rational, its denominator being N, the order of the singularities. Conversely, two monodromic solutions imply an apparent singularity.

It is clear that $(Ay_1^N + By_2^N)$ is a monodromic product, and is an *irreducible* solution of our problem for $n = N$. Expressions such as $y_1(Ay_1^N + By_2^N)$ or $y_1 y_2(Ay_1^N + By_2^N)$ are also monodromic solutions for $n = N+1$ and $n = N+2$ respectively. But as they break up into distinct groups of factors, which are separately monodromic, they may be termed *reducible*, and will be left aside as trivial.

9.1. When there are two quadratic singularities, it was seen in (4.2) that, instead of Gauss's form, we could use the equation of Elliot and Darboux,

$$\frac{d^2y}{d\theta^2} = [m(m+1)\operatorname{cosec}^2\theta + n(n+1)\sec^2\theta + k^2]y,$$

where the independent variable of Gauss's equation is $x = \sin^2\theta$, and the exponent differences are $m+\frac{1}{2}$, $n+\frac{1}{2}$, and ik, m and n being integers. Thus ik is a real fraction, whose denominator N is the order of the third singularity in the symbol $(2, 2, N)$.

The solutions can be written

$$y_1 = e^{k\theta}R(\cot\theta), \qquad y_2 = e^{-k\theta}R(-\cot\theta),$$

where the symbol R represents a rational function, the only poles of $R(z)$ being $z = 0$ and $z = \infty$.

The conditions for monodromy at the points $x = 0$, $x = \infty$, and $x = 1$ of Gauss's form, correspond to the conditions that our product (which is a homogeneous polynomial in y_1, y_2) shall be monodromic, when θ is replaced by $-\theta$ or by $\theta+\pi$.

The polynomial must accordingly be monodromic, when (y_1, y_2) are replaced by (y_2, y_1) or by $(\omega y_1, \omega^{-1}y_2)$, where $\omega = e^{k\pi}$ and $\omega^N = -1$. After verifying that the polynomial satisfies the conditions, we examine whether it breaks up into factors, which are separately monodromic. Such solutions would be *reducible* and we have agreed to ignore them. We tabulate the irreducible solutions, leaving the verification to the reader.

9.2. If $n = 4$, we have $N \leqslant 4$ by (5.3). The first substitution separates the two types of solution

$$A(y_1^4 + y_2^4) + By_1 y_2(y_1^2 + y_2^2) + Cy_1^2 y_2^2,$$

$$(y_1^2 - y_2^2)[A(y_1^2 + y_2^2) + By_1 y_2].$$

The second substitution yields, after eliminating the reducible cases:

$$N=1,\qquad A(y_1^4+y_2^4)+By_1y_2(y_1^2+y_2^2)+Cy_1^2y_2^2,$$

$$N=2,\,3,\qquad \text{no irreducible solution,}$$

$$N=4,\qquad A(y_1^4+y_2^4),\qquad A(y_1^4-y_2^4).$$

9.3. If $n=5$, we must have $N\leqslant 5$.

The first substitution separates the solutions

$$A(y_1^5+y_2^5)+By_1y_2(y_1^3+y_2^3)+Cy_1^2y_2^2(y_1+y_2),$$

$$A(y_1^5-y_2^5)+By_1y_2(y_1^3-y_2^3)+Cy_1^2y_2^2(y_1-y_2).$$

The second substitution yields no irreducible solutions for $N=1$, 2, 3, 4. If $N=5$, the irreducible solutions are

$$A(y_1^5+y_2^5),\qquad A(y_1^5-y_2^5).$$

9.4. The discussion of this and the preceding section has disposed of the cases, where the symbol (N_1, N_2, N_3) contains unity, or where 2 is repeated. We need only attend to those symbols with no figure less than 2, and with two figures not less than 3. Any admissible equation falls under one or more of the following types, which will now be dealt with:

$$\begin{array}{llll} n=4, & (4,4,N), & (4,3,N), & (3,3,N); \\ n=5, & (5,5,N), & (5,4,N), & (5,3,N), \\ & (4,4,N), & (4,3,N), & (3,3,N). \end{array}$$

10.1. It was shown in (5.2) that, if y_1, y_2 are the fundamental solutions at a singularity of order N, the factors y_1, y_2 may occur singly in our monodromic product, but all other factors occur in groups $(Ay_1^N+By_2^N)$. Absorbing the coefficients, real or complex, into the solutions, the canonical forms for $n=4$ are $(P_{\alpha_1}^4+P_{\alpha_2}^4)$ if $N=4$, and $P_{\alpha_1}(P_{\alpha_1}^3+P_{\alpha_2}^3)$ if $N=3$. Thus a solution of the type $(4,4,N)$ or $(4,3,N)$ would lead to one or other of the identities

$$P_{\alpha_1}^4+P_{\alpha_2}^4\equiv P_{\gamma_1}^4+P_{\gamma_2}^4,\qquad P_{\alpha_1}^4+P_{\alpha_2}^4\equiv P_{\gamma_1}(P_{\gamma_1}^3+P_{\gamma_2}^3),$$

respectively. If we proceed as in (6.2), putting

$$P_{\alpha_1}=pP_{\gamma_1}+qP_{\gamma_2},\qquad P_{\alpha_2}=rP_{\gamma_1}+sP_{\gamma_2},\qquad ps-qr\neq 0,$$

and comparing coefficients, it is found that the second identity leads to no solution, and that the first can be reduced, by proper choice of notation, to the case $P_{\alpha_1}=P_{\gamma_1}$, $P_{\alpha_2}=P_{\gamma_2}$, which is covered by the discussion in (8).

10.2. In $(3,3,N)$, we use Gauss's form and we observe that a, b, c are *real*, because the exponent-differences are real. There is no loss

of generality in assuming that the singularities of the third order are at $x=0$ and $x=1$, and we can arrange that the branches P_{α_1}, P_{α_2}, P_{γ_1}, P_{γ_2} shall be real, if x is real and $0<x<1$. We have, in that interval, the identity

$$P_{\alpha_1}(AP_{\alpha_1}^3+BP_{\alpha_2}^3)\equiv P_{\gamma_1}(CP_{\gamma_1}^3+DP_{\gamma_2}^3),$$

whose coefficients might perhaps be complex. Now, by (2.3), P_{α_1} is identifiable with one of the linear factors on the right, and for an irreducible solution, this must not be P_{γ_1}. Hence there must be another real linear factor, so that C/D is real, and similarly A/B is real. Absorbing a *real* coefficient into each fundamental solution, we find

$$P_{\alpha_1}(P_{\alpha_1}^3+P_{\alpha_2}^3)\equiv \Theta P_{\gamma_1}(P_{\gamma_1}^3+P_{\gamma_2}^3).$$

The coefficient Θ is the ratio of two real functions, and is therefore real, and may be taken as ± 1, without loss of generality. We now have

$$P_{\alpha_1}=l(P_{\gamma_1}+P_{\gamma_2}),\qquad P_{\gamma_1}=m(P_{\alpha_1}+P_{\alpha_2}),\tag{10.21}$$

where l, m are real and different from zero. Hence

$$l(P_{\alpha_1}^2-P_{\alpha_1}P_{\alpha_2}+P_{\alpha_2}^2)\equiv \pm m(P_{\gamma_1}^2-P_{\gamma_1}P_{\gamma_2}+P_{\gamma_2}^2),$$
$$l^3(3m^2P_{\alpha_1}^2-3mP_{\alpha_1}P_{\gamma_1}+P_{\gamma_1}^2)\equiv \pm m^3(P_{\alpha_1}^2-3lP_{\alpha_1}P_{\gamma_1}+3l^2P_{\gamma_1}^2).$$

The coefficients of $P_{\alpha_1}P_{\gamma_1}$ give $l^2=\pm m^2$, but the lower sign leads to imaginary values, and must be rejected. Our identity now requires

$$l^2=m^2,\qquad l=\pm m,\qquad 3lm=1,$$

but again the lower sign leads to imaginary values, so that we have

$$l=m=\pm 1/\surd 3.$$

These two forms differ only in notation, so we may take the upper sign. We must now verify that the formulae

$$\surd 3P_{\alpha_1}=P_{\gamma_1}+P_{\gamma_2},\qquad \surd 3P_{\gamma_1}=P_{\alpha_1}+P_{\alpha_2},\tag{10.22}$$

are compatible with the classical theory of the hypergeometric equation. On removing the factor $x^{\alpha_1}(1-x)^{\gamma_1}$, the branches P_{α_1} and P_{γ_1} are identified with those of exponent zero at $x=0$ and at $x=1$ respectively. We consider the real interval $0<x<1$, and, to keep the formulae as simple as possible, we use the notation:*

$$\left.\begin{aligned}
&P_{\alpha_1}\propto Y_1\equiv F(a,b;c;x),\\
&P_{\alpha_2}\propto Y_2\equiv x^{1-c}F(a-c+1,b-c+1;2-c;x),\\
&P_{\gamma_1}\propto Y_3\equiv F(a,b;a+b-c+1;1-x),\\
&P_{\gamma_2}\propto Y_4\equiv (1-x)^{c-a-b}F(c-a,c-b;c-a-b+1;1-x).
\end{aligned}\right\}\tag{10.23}$$

* Cf. Forsyth, *Differential Equations*, Ch. VI.

Now the formulae (10.22) must be equivalent to

$$\left.\begin{aligned} Y_1 &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} Y_3 + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} Y_4, \\ Y_3 &= \frac{\Gamma(1-c)\Gamma(1+a+b-c)}{\Gamma(1+a-c)\Gamma(1+b-c)} Y_1 + \frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)} Y_2. \end{aligned}\right\} \quad (10.24)$$

On comparing these two sets of formulae, we find:

$$\frac{\sqrt{3}P_{\alpha_1}}{P_{\gamma_1}} = \frac{\Gamma(c-a)\,\Gamma(c-b)\,Y_1}{\Gamma(c)\,\Gamma(c-a-b)\,Y_3}, \qquad \frac{\sqrt{3}P_{\gamma_1}}{P_{\alpha_1}} = \frac{\Gamma(1+a-c)\Gamma(1+b-c)\,Y_3}{\Gamma(1-c)\Gamma(1+a+b-c)\,Y_1},$$

hence

$$3 = \frac{\Gamma(c-a)\Gamma(1+a-c)\Gamma(c-b)\Gamma(1+b-c)}{\Gamma(c)\Gamma(1-c)\Gamma(c-a-b)\Gamma(1+a+b-c)} = \frac{\sin\pi c\sin\pi(c-a-b)}{\sin\pi(c-a)\sin\pi(c-b)}. \quad (10.25)$$

In this equation, we observe that, if a, b, c is any solution, another can be found by adding any integers to a, b, c. We shall therefore fix our attention on the solutions for which a, b, c are positive proper fractions, which will be called the *reduced* solutions. Since a, b occur symmetrically, we may take

$$1 > a \geqslant b \geqslant 0, \qquad 1 > c \geqslant 0.$$

We may note that the hypergeometric function with $a = b$ would itself be logarithmic, and so could not lead to a solution of our problem; but an equation whose *reduced* form was of this type might have an apparent singularity. Actually, the case does not arise, as $(a-b)$ is found not to be an integer. Now, since $x = 0$ and $x = 1$ are singularities of the third order, we must have

$$c = \tfrac{1}{3} \text{ or } \tfrac{2}{3}, \qquad c-a-b = p \pm \tfrac{1}{3},$$

where p is an integer. If these expressions have the same fractional part, so that $(a+b)$ is an integer, (10.25) gives

$$3\sin\pi(c-a)\sin\pi(c-b) = \cos\pi(a+b)\sin^2\pi c = \tfrac{3}{4}\cos\pi(a+b),$$

$$2\cos\pi(a-b) - 2\cos\pi(2c-a-b) = \cos\pi(a+b),$$

$$2\cos\pi(a-b) = \cos\pi(a+b)[1+2\cos 2\pi c] = 0.$$

The solution requires, with our conventions,

$$a-b = \tfrac{1}{2}, \qquad a+b = 1, \qquad c = \tfrac{1}{3} \text{ or } \tfrac{2}{3},$$

and the corresponding reduced forms belong to

$$F(\tfrac{3}{4}, \tfrac{1}{4}; \tfrac{1}{3}; x) \text{ or } F(\tfrac{3}{4}, \tfrac{1}{4}; \tfrac{2}{3}; x). \quad (10.26)$$

If $(a+b)$ is *not* an integer, $(2c-a-b)$ *is* an integer, and we can then put (10.25) in the form

$$3\sin\pi(c-a)\sin\pi(c-b) = -\sin^2\pi c\cos\pi(2c-a-b)$$
$$= -\tfrac{3}{4}\cos\pi(2c-a-b),$$
$$2\cos\pi(a-b) = \cos\pi(2c-a-b) = \pm 1.$$

Taking the upper sign, our conventions give the reduced solution

$$a-b = \tfrac{1}{3}, \qquad a+b = 2c, \qquad c = \tfrac{1}{3} \text{ or } \tfrac{2}{3},$$

and the hypergeometric functions are derivable from

$$F(\tfrac{1}{2},\tfrac{1}{6};\tfrac{1}{3};x) \text{ or } F(\tfrac{5}{6},\tfrac{1}{2};\tfrac{2}{3};x). \tag{10.27}$$

If we choose the lower sign, $(2c-a-b) = \pm 1$, and $(a-b) = \tfrac{2}{3}$, and there are *no* reduced solutions.

The exponent-difference at $x = \infty$ being $(a-b)$, we see that this point is of the second order in (10.26), and of the third order in (10.27); the symbols of the families are $(3,3,2)$ and $(3,3,3)$ respectively, although not every hypergeometric equation with these symbols belongs to our families. For example, if $a = \tfrac{1}{3}$, $b = 0$, $c = \tfrac{2}{3}$, all the exponent-differences are $\tfrac{1}{3}$, but the equation is not admissible.

If a, b, c differ by integers from a set of reduced values, we shall have, in the region where $|x| < 1$, $|1-x| < 1$,

$$F(x) \equiv P_{\alpha_1}(P^3_{\alpha_1}+P^3_{\alpha_2}) \equiv P_{\gamma_1}(P^3_{\gamma_1}+P^3_{\gamma_2}).$$

At $x = 0$ and $x = 1$, $F(x)$ remains uniform, and is either regular or has poles at those points, according as the integers $3(1-c)$ and $3(a+b-c)$ are positive or negative. It has no other singularities in the finite part of the plane, and so must be a uniform analytic function which has at most a pole at infinity; it is therefore a rational function of x. Equivalent forms of $F(x)$ include

$$\sqrt{3}P_{\alpha_1}P_{\gamma_1}(P^2_{\alpha_1}-P_{\alpha_1}P_{\alpha_2}+P^2_{\alpha_2}), \qquad \sqrt{3}P_{\alpha_1}P_{\gamma_1}(P^2_{\gamma_1}-P_{\gamma_1}P_{\gamma_2}+P^2_{\gamma_2}),$$
$$3\sqrt{3}P_{\alpha_1}P_{\gamma_1}(P^2_{\alpha_1}-\sqrt{3}P_{\alpha_1}P_{\gamma_1}+P^2_{\gamma_1}), \qquad 3\sqrt{3}P_{\alpha_1}P_{\gamma_1}(P^2_{\alpha_1}-P_{\gamma_1}P_{\gamma_2}),$$
$$3\sqrt{3}P_{\alpha_1}P_{\gamma_1}(P^2_{\gamma_1}-P_{\alpha_1}P_{\alpha_2}).$$

If we pass from these to the classical hypergeometric functions, we can find a number of equivalent forms, which are multiples of $F(x)$, such as

$$Y_1Y_3\left(\frac{\Gamma(1-c)\Gamma(1+a+b-c)}{\Gamma(1+a-c)\Gamma(1+b-c)}Y_1^2-Y_1Y_3+\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}Y_3^2\right),$$

which is a rational function of x.

11.1. We now show that none of the six types of hypergeometric equation enumerated in (9.4) can yield a solution of our problem for $n=5$. For the first three types, a formal argument following the lines of (6.2) or (10.1) will show that the identities

$$\begin{aligned}
&(5,5,N) \qquad P^5_{\alpha_1}+P^5_{\alpha_2}\equiv P^5_{\gamma_1}+P^5_{\gamma_1},\\
&(5,4,N) \qquad P^5_{\alpha_1}+P^5_{\alpha_2}\equiv P_{\gamma_1}(P^4_{\gamma_1}+P^4_{\gamma_2}),\\
&(5,3,N) \qquad P^5_{\alpha_1}+P^5_{\alpha_2}\equiv P_{\gamma_1}P_{\gamma_2}(P^3_{\gamma_1}+P^3_{\gamma_2}),
\end{aligned}$$

yield only the type (5, 5, 1), covered by (8).

11.2. In the other cases, the reader will argue, as in (10.2), that, in the real interval $0<x<1$, we have *real* identities connecting the four *real* solutions P_{α_1}, P_{α_2}, P_{γ_1}, P_{γ_2}. The proof then proceeds as follows, in the three cases.

11.3. For $(4,4,N)$, the real identity is

$$P_{\alpha_1}(P^4_{\alpha_1}-P^4_{\alpha_2})\equiv P_{\gamma_1}(P^4_{\gamma_1}-P^4_{\gamma_2}).$$

For an irreducible solution, the three real factors must correspond in a way, which (except in notation) can only be as follows:

$$P_{\alpha_1}=l(P_{\gamma_1}+P_{\gamma_2}),\quad P_{\gamma_1}=m(P_{\alpha_1}+P_{\alpha_2}),\quad (P_{\alpha_1}-P_{\alpha_2})\propto(P_{\gamma_1}-P_{\gamma_2}).$$

We find $4lm=1$, so that we put $\rho=2l=1/2m$, and obtain

$$\begin{gathered}
2P_{\alpha_1}=\rho(P_{\gamma_1}+P_{\gamma_2}),\qquad 2P_{\alpha_2}=\rho(3P_{\gamma_1}-P_{\gamma_2}),\\
\rho^3(P^2_{\alpha_1}+P^2_{\alpha_2})+(P^2_{\gamma_1}+P^2_{\gamma_2})\equiv 0,\\
\rho^5(5P^2_{\gamma_1}-2P_{\gamma_1}P_{\gamma_2}+P^2_{\gamma_2})+2(P^2_{\gamma_1}+P^2_{\gamma_2})\equiv 0,
\end{gathered}$$

which is impossible, when P_{γ_1}, P_{γ_2} are independent.

11.4. For $(4,3,N)$ the real identity is

$$P_{\alpha_1}(P^4_{\alpha_1}-P^4_{\alpha_2})\equiv P_{\gamma_1}P_{\gamma_2}(P^3_{\gamma_1}+P^3_{\gamma_2}),$$

and the correspondence of factors, for an irreducible solution, can always be written

$$P_{\gamma_1}=l(P_{\alpha_1}+P_{\alpha_2}),\qquad P_{\gamma_2}=m(P_{\alpha_1}-P_{\alpha_2}),\qquad P_{\alpha_1}\propto(P_{\gamma_1}+P_{\gamma_2}).$$

We have $l=m$, and reduce the identity to

$$(P^2_{\alpha_1}+P^2_{\alpha_2})\equiv 2l^3(P^2_{\gamma_1}-P_{\gamma_1}P_{\gamma_2}+P^2_{\gamma_2})\equiv 2l^5(P^2_{\alpha_1}+3P^2_{\alpha_2}),$$

which is impossible, when P_{α_1}, P_{α_2} are independent.

11.5. For $(3,3,N)$, the real identity is

$$P_{\alpha_1}P_{\alpha_2}(P^3_{\alpha_1}+P^3_{\alpha_2})\equiv P_{\gamma_1}P_{\gamma_2}(P^3_{\gamma_1}+P^3_{\gamma_2}),$$

and an irreducible solution would require that P_{α_1} and P_{α_2} should be proportional to two distinct *real* factors of $P^3_{\gamma_1}+P^3_{\gamma_2}$, which is impossible.

A PROBLEM IN ADDITIVE ARITHMETIC

By LEONARD CARLITZ

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Introduction

THIS paper is concerned with the following problem.

Let $\alpha(n)$ *be a function of the positive integral variable* n *satisfying the following conditions*:*

$$\left.\begin{aligned} &\alpha(1)=1, \qquad \alpha(n)=O(n^{\epsilon}), \qquad \alpha(p)=1+O(p^{-\frac{1}{2}+\epsilon}), \\ &\alpha(mn)=\alpha(m)\alpha(n) \text{ for } (m,n)=1, \end{aligned}\right\} \tag{1}$$

where p *is any prime, and* ϵ *is an arbitrary positive quantity. Then we seek an asymptotic expression for*

$$R_\nu(n)=\sum \alpha(n_1)\alpha(n_2)\ldots\alpha(n_\nu) \qquad (\nu \geqslant 3),$$

the sum being taken over all the partitions of n *into* ν *positive integers.*

The method used is that of Hardy and Littlewood, particularly that as applied to the Goldbach problem,† and simplified by Landau in his *Vorlesungen über Zahlentheorie*.‡ It should be remarked, however, that it is here not necessary to introduce any assumptions relative to the zeros of the Dirichlet L-functions. Our main result is the following:

$$\left| R_\nu(n)-\frac{b^\nu n^{\nu-1}}{(\nu-1)!}S(n)\right| < An^{\nu-\frac{3}{2}+\epsilon}, \tag{2}$$

where $$b=\prod_p \left(1+\frac{\alpha(p)-1}{p}+\frac{\alpha(p^2)-\alpha(p)}{p^2}+\ldots\right);$$

$$S(n)=\prod_p \left(1-\frac{\tau^\nu(p)}{p^\nu}\right)\times$$
$$\times \prod_{p|n} \frac{1+\dfrac{p^{-1}}{p^\nu}\tau^\nu(p)+\ldots+\dfrac{p^f-p^{f-1}}{p^{f\nu}}\tau^\nu(p^f)+\dfrac{-p^f}{p^{(f+1)\nu}}\tau^\nu(p^{f+1})}{1-\dfrac{\tau^\nu(p)}{p^\nu}}, \tag{3}$$

p^f being the highest power of p dividing n;

* Slightly less stringent conditions on $\alpha(n)$ would suffice. Thus the condition on p need not hold for a finite number of primes; also the 1/2 in the exponent might be decreased somewhat.

† 'Some Problems of Partitio Numerorum, III: On the Expression of a Number as a Sum of Primes': *Acta Math.* 44 (1923), 1–70.

‡ Bd. i, 5. Teil, pp. 183–234. Cited as *Vorlesungen*, i.

$$\tau(p^e) = \frac{\alpha(p^e)-\alpha(p^{e-1}) + \dfrac{\alpha(p^{e+1})-\alpha(p^e)}{p} + \ldots}{1 + \dfrac{\alpha(p)-1}{p} + \ldots}; \tag{4}$$

and A is a positive quantity depending only on ϵ.

If we suppose that, in addition to the restrictions in (1), $\alpha(n)$ is always either zero or one, then (2) gives an asymptotic expression for the number of representations of a large integer n by ν numbers n_i for each of which $\alpha(n_i) = 1$. Such an application will be considered in § 5. We also give one other application of the general formula (2) (§ 6); it is easy to see how many other applications may be made.

1. Transformation of the generating function

We are going to follow rather closely the analysis in Part V of *Vorlesungen*, i. We begin with the function

$$f(x) = \sum_{m=1}^{\infty} \alpha(m) x^m, \tag{5}$$

which is evidently convergent inside the unit circle of the x-plane. If G denote a circle with centre at the origin and of radius $e^{-1/n}$, then, of course,

$$R_\nu(n) = \frac{1}{2\pi i} \int_G \frac{f^\nu(x)}{x^{n+1}}\, dx. \tag{6}$$

Consider the Farey* series l/k,

$$0 < k \leqslant n^{\frac{1}{2}}, \qquad 0 \leqslant 1 \leqslant k, \qquad (l, k) = 1.$$

The circle G is divided into arcs B_ρ by means of the points

$$\exp\left(-\frac{1}{n} + 2\pi i \frac{l+l'}{k+k'}\right),$$

l/k and l'/k' being two adjacent fractions in the series. Each arc contains just one point $e^{-1/n}\rho$ $(\rho = e^{2l\pi i/k})$; on B_ρ put

$$x = \rho X, \qquad X = e^{-y}, \qquad y = \frac{1}{n} - \theta i,$$

so that

$$-\theta_1 \leqslant \theta \leqslant \theta_2, \qquad \frac{\pi}{kn^{1/2}} \leqslant \theta_j < \frac{2\pi}{kn^{1/2}}, \qquad (j = 1, 2).$$

We shall replace $f(x)$ on B_ρ by

$$\psi_\rho(x) = \frac{b}{ky} \tau(k),$$

* Cf. *Vorlesungen*, i, p. 217.

where $\tau(k)$ is defined by (4) and

$$\tau(k) = \prod_{p|k} \tau(p^e), \qquad k = \prod_{p|k} p^e.$$

Our object is to prove

$$|f(x) - \psi_\rho(x)| < An^{7/8+\epsilon}. \tag{7}$$

By Mellin's Formula,* for $R(y) > 0$,

$$e^{-y} = \frac{1}{2\pi i} \int_{(\beta)} y^{-s}\Gamma(s)\, ds,$$

where

$$\int_{(\beta)} = \int_{\beta-\infty i}^{\beta+\infty i} \qquad (\beta \text{ real}).$$

Now

$$\begin{aligned} f(x) &= \sum \alpha(m)\rho^m e^{-ym} \\ &= \frac{1}{2\pi i} \sum \alpha(m)\rho^m \int_{(2)} (my)^{-s}\Gamma(s)\, ds. \\ &= \frac{1}{2\pi i} \int_{(2)} y^{-s}\Gamma(s)\left(\sum \frac{\alpha(m)\rho^m}{m^s}\right) ds.\dagger \end{aligned} \tag{8}$$

But

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{\alpha(m)\rho^m}{m^s} &= \sum_{d|k} \sum_{(m,k)=d} \frac{\alpha(m)\rho^m}{m^s} \\ &= \sum_{k=d\delta} \frac{1}{d^s} \sum_{(m,\delta)=1} \frac{\alpha(dm)\rho^{dm}}{m^s} \\ &= \sum_{k=d\delta} \frac{1}{d^s} \sum_{(m,\delta)=1} \frac{\alpha(dm)\rho^m(\delta)}{m^s}, \end{aligned} \tag{9}$$

$\rho(\delta)$ denoting a primitive δth root of unity. If we put

$$B(\chi) = \sum_{m=1}^{\delta} \chi(m)\rho^m,$$

χ being a character (mod δ), the sum in (9) becomes

$$\sum_{k=d\delta} \frac{1}{d^s} \frac{1}{\phi(\delta)} \sum_{\chi} B(\chi^{-1}) \sum_{m=1}^{\infty} \frac{\alpha(dm)\chi(m)}{m^s}. \tag{10}$$

* Mellin, 'Abriss einer einheitlichen Theorie der Gamma- und hypergeometrischen Functionen': *Math. Annalen*, 68 (1910), 305–37.

† The interchange of integration and summation may be justified without difficulty; cf. *Vorlesungen*, i, p. 213.

Now letting $d = \Pi q^f$,

$$\sum \frac{\alpha(dm)\chi(m)}{m^s}$$

$$= \prod_{p \nmid d}\left(1+\frac{\alpha(p)\chi(p)}{p^s}+\frac{\alpha(p^2)\chi(p^2)}{p^{2s}}+\ldots\right)\prod_{q|d}\left(\alpha(q^f)+\frac{\alpha(q^{f+1})\chi(q)}{q^s}+\ldots\right)$$

$$= \prod_{\text{all } p}\left(1+\frac{\alpha(p)\chi(p)}{p^s}+\ldots\right)\times$$

$$\times \prod_{q|d}\left(\alpha(q^f)+\frac{\alpha(q^{f+1})\chi(q)}{q^s}+\ldots\right)\left(1+\frac{\alpha(q)\chi(q)}{q^s}+\ldots\right)^{-1}$$

$$= \eta(s,\chi)\,\xi(s,\chi,d), \text{ say.} \tag{11}$$

As is customary, put

$$L(s,\chi) = \sum_{m=1}^{\infty}\frac{\chi(m)}{m^s} = \prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1};$$

then

$$\frac{\eta(s,\chi)}{L(s,\chi)} = \prod_p\left(1+\frac{\alpha(p)\chi(p)}{p^s}+\ldots\right)\left(1-\frac{\chi(p)}{p^s}\right)$$

$$= \prod_p\left(1+\frac{\alpha(p)-1}{p^s}\chi(p)+\frac{\alpha(p^2)-\alpha(p)}{p^{2s}}\chi(p^2)+\ldots\right).$$

Using (1) we have

$$\left|\frac{\eta(s,\chi)}{L(s,\chi)}\right| < \prod\left(1+\frac{C}{p^{\sigma+1/2}}+\frac{C}{p^{2\sigma}(1-p^{-\sigma})}\right)$$

$$< A = A(\epsilon) \text{ for } \sigma \geqslant \tfrac{1}{2}+\epsilon. \tag{12}$$

We see, then, that in the half-plane $\sigma > \frac{1}{2}$, $\eta(s,\chi)$ is regular except for a pole of order one at $s=1$ when $\chi = \chi_0$, the principal character. The residue at this pole is by (12)

$$\prod_{p|\delta}\left(1-\frac{1}{p}\right)\prod_{p\nmid\delta}\left(1+\frac{\alpha(p)-1}{p}+\ldots\right) = \frac{b\phi(\delta)}{\delta\psi(\delta)}, \tag{13}$$

where

$$\psi(\delta) = \prod_{p|\delta}\left(1+\frac{\alpha(p)-1}{p}+\ldots\right).$$

Returning to (8), we deduce from

$$\sum\frac{\alpha(m)\rho^m}{m^s} = \sum_{k=d\delta}\frac{1}{d^s}\frac{1}{\phi(\delta)}\sum_{\chi}B(\chi^{-1})\eta(s,\chi)\xi(s,\chi,d) \tag{14}$$

that the integrand has a pole of order one at $s=1$; using (13), the residue at this pole is

$$\frac{1}{y}\sum_{k=d\delta}\frac{1}{d\phi(\delta)}B(\chi_0)\frac{b\phi(\delta)}{\delta\psi(\delta)}\xi(1,\chi_0,d) \tag{15}$$

$$=\frac{b}{ky}\sum_{k=d\delta}\mu(\delta)\frac{\xi(1,\chi_0,d)}{\psi(\delta)}*$$

$$=\frac{b}{ky}\prod_{p|k}\sum_{p^f=d\delta}\mu(\delta)\frac{\xi(1,\chi_0,d)}{\psi(\delta)}$$

$$=\frac{b}{ky}\prod_{p|k}\left\{\xi(1,\chi_0(1),p^f)-\frac{\xi(1,\chi_0(p),p^{f-1})}{\psi(p)}\right\}$$

$$=\frac{b}{ky}\prod_{p|k}\left\{\frac{\sum_e\frac{\alpha(p^{e+f})}{p^e}}{\sum_e\frac{\alpha(p^e)}{p^e}}-\frac{-\alpha(p^{f-1})}{1+\frac{\alpha(p)-1}{p}+\dots}\right\}$$

$$=\frac{b}{ky}\tau(k)=\psi_\rho(x). \tag{16}$$

Returning again to (8), we consider the integral of

$$y^{-s}\Gamma(s)\sum\frac{\alpha(m)\rho^m}{m^s}$$

taken in the positive direction around the rectangle of vertices

$$2\pm Mi,\qquad \tfrac{1}{2}+\epsilon\pm Mi,\qquad (M>0).$$

From the results above the integrand is uniform within and on the rectangle and is regular except for the pole at $s=1$ with residue as given by (16). Furthermore, exactly as in *Vorlesungen*, i, p. 215, it may be shown that the integral along $\sigma=\frac{1}{2}+\epsilon$ converges and that the integrals along the horizontal lines approach zero when M becomes infinite. Accordingly

$$\frac{1}{2\pi i}\int_{(2)}=\psi_\rho(x)+\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)}. \tag{17}$$

We must now find an upper bound for the integral in the right member of (17); this requires a lemma on the magnitude of $L(s,\chi)$ on the line $\sigma=\frac{1}{2}+\epsilon$.

* $\mu(\delta)$ is the Möbius μ-function; it appears here as $B(\chi)$.

2. Magnitude of $L(s,\chi)$ in the critical strip

LEMMA. *On* $\sigma=\frac{1}{2}+\epsilon$, $(\epsilon>0;\ \chi, \bmod k)$,

$$|L(s,\chi)|<Ak^{\frac{1}{2}+\epsilon}(|t|+1)^{\frac{1}{2}+\epsilon}. \tag{18}$$

Proof. We may suppose χ a primitive character (mod k). For if it is imprimitive, there exists* a K, $K|k$, and a primitive character X (mod K) such that

$$L(s,\chi)=\prod_{p|k}\left(1-\frac{X(p)}{p^s}\right)L(s,X).$$

Then on $\sigma=\frac{1}{2}+\epsilon$,

$$\begin{aligned}|L(s,\chi)| &\leqslant |L(s,X)|\prod_{p|k}(1+p^{-\frac{1}{2}-\epsilon})\\ &<|L(s,X)|\prod_{p|k}2<Ak^{\epsilon}|L(s,X)|.\end{aligned}$$

Hence, by properly modifying ϵ, if (18) be proved for all primitive characters it will hold generally.

If now, χ being primitive, we let

$$\Xi(s,\chi)=\left(\frac{\pi}{k}\right)^{-\frac{s+a}{2}}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi),$$

where $a=0$ for $\chi(-1)=1$, $a=1$ for $\chi(-1)=-1$, then†

$$\Xi(s,\chi)=\mathrm{H}(\chi)\Xi(1-s,\chi^{-1}),\qquad |\mathrm{H}(\chi)|=1. \tag{19}$$

Let $s=-\epsilon+ti$, so that

$$|L(1-s,\chi^{-1})|\leqslant A(\epsilon). \tag{20}$$

From (19),

$$\left(\frac{\pi}{k}\right)^{-\frac{s+a}{2}}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi)=\mathrm{H}(\chi)\left(\frac{\pi}{k}\right)^{-\frac{1-s+a}{2}}\Gamma\left(\frac{1-s+a}{2}\right)L(1-s,\chi^{-1}).$$

But

$$\begin{aligned}\left|\frac{\Gamma\left(\frac{1-s+a}{2}\right)}{\Gamma\left(\frac{s+a}{2}\right)}\right| &=\left|\frac{\Gamma\left(\frac{1+\epsilon+a-ti}{2}\right)\Gamma\left(\frac{2+\epsilon-a-ti}{2}\right)}{\Gamma\left(\frac{a-\epsilon+ti}{2}\right)\Gamma\left(\frac{2+\epsilon-a-ti}{2}\right)}\right|\\ &\leqslant e^{\pi|t|}A(\epsilon)e^{-\frac{\pi}{2}|t|}(|t|+1)^{\frac{1+\epsilon+a}{2}-\frac{1}{2}}A(\epsilon)e^{-\frac{\pi}{2}|t|}(|t|+1)^{\frac{2+\epsilon-a}{2}-\frac{1}{2}}\\ &<A(|t|+1)^{\frac{1}{2}+\epsilon}.\ddagger\end{aligned}$$

Hence

$$\begin{aligned}|L(s,\chi)| &\leqslant\left(\frac{k}{\pi}\right)^{\frac{1+\epsilon+a}{2}-\frac{-\epsilon+a}{2}}A(|t|+1)^{\frac{1}{2}+\epsilon}\\ &<Ak^{\frac{1}{2}+\epsilon}(|t|+1)^{\frac{1}{2}+\epsilon}.\end{aligned} \tag{21}$$

* Cf. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, i (1909), pp. 478 ff.

† Landau, loc. cit., p. 497.

‡ Cf. *Vorlesungen*, i, p. 195.

We now apply an extension of the Phragmén-Lindelöf Theorem, which may be enunciated thus.*

Let $\beta > \alpha$, $f(s)$ be regular in the strip $\alpha \leqslant \sigma \leqslant \beta$;

$$|f(s)| < C_1(|t|+1)^a \text{ for } \sigma = \alpha,$$
$$|f(s)| < C_1(|t|+1)^b \text{ for } \sigma = \beta,$$
$$|f(s)| < C_2(|t|+1)^{C_3} \text{ for } \alpha < \sigma < \beta,$$

where C_1, C_2, C_3 are independent of s, a and b absolute constants. Then for $\alpha \leqslant \sigma \leqslant \beta$

$$|f(s)| < 2C_1(|t|+1)^{a\frac{\beta-\sigma}{\beta-\alpha}+b\frac{\sigma-\alpha}{\beta-\alpha}}.$$

For the application, let

$$\alpha = 1+\epsilon, \qquad \beta = -\epsilon;$$
$$a = 0, \qquad b = \tfrac{1}{2}+\epsilon;$$
$$f(s) = L(s,\chi), \qquad \chi \neq \chi_0.$$

In view of (20) and (21) and the easily-proved inequality

$$|L(s,\chi)| < C_4(|t|+1)^{C_5} \qquad (\alpha < \sigma < \beta),$$

it follows that the theorem applies, and therefore, on $\sigma = \frac{1}{2}+\epsilon$,

$$|L(s,\chi)| < Ak^{\frac{1}{2}+\epsilon}(|t|+1)^{\frac{1}{4}+\epsilon}. \tag{22}$$

By using $f(s) = k^{-s/2}L(s,\chi)$, (22) can be replaced by

$$|L(s,\chi)| < Ak^{\frac{1}{4}+\epsilon}(|t|+1)^{\frac{1}{4}+\epsilon}. \tag{18}$$

For $\chi = \chi_0$, remembering that χ is primitive, and that therefore $K = 1$,

$$|L(s,\chi)| = |\zeta(s)| \leqslant A(|t|+1)^{\frac{1}{4}+\epsilon} \leqslant Ak^{\frac{1}{4}+\epsilon}(|t|+1)^{\frac{1}{4}+\epsilon}.$$

3. Proof of (7)

We now return to the right member of (17). On $\sigma = \frac{1}{2}+\epsilon$,

$$|y^{-s}| \leqslant |y|^{-\frac{1}{2}-\epsilon}e^{|t|.|\operatorname{arc} y|};$$
$$|\Gamma(s)| < Ae^{-\frac{1}{2}\pi|t|}(|t|+1)^{\epsilon};$$
$$|\eta(s,\chi)| = |L(s,\chi)|.\left|\frac{\eta(s,\chi)}{L(s,\chi)}\right| < Ak^{\frac{1}{4}+\epsilon}(|t|+1)^{\frac{1}{4}+\epsilon},$$

by (18) and (12);

$$|\xi(s,\chi,d)| < Ad^{\epsilon} \leqslant Ak^{\epsilon}.$$

* Cf. *Vorlesungen*, ii, p. 49.

Accordingly, by (14),

$$\left|\int\limits_{(\frac{1}{2}+\epsilon)} y^{-s}\Gamma(s)\sum\frac{\alpha(m)\rho^m}{m^s}\,ds\right|$$

$$\leqslant A\sum_{k=d\delta}\frac{1}{\phi(\delta)}\sum_{\chi}|B(\chi^{-1})|\cdot\left|\int\limits_{(\frac{1}{2}+\epsilon)} y^{-s}\Gamma(s)\eta(s,\chi)\xi(s,\chi,d)\,ds\right|, \tag{23}$$

$$\left|\int\limits_{(\frac{1}{2}+\epsilon)} y^{-s}\Gamma(s)\eta(s,\chi)\xi(s,\chi,d)\,ds\right| < Ak^{\frac{1}{4}+\epsilon}|y|^{-\frac{1}{2}-\epsilon}\int\limits_0^\infty e^{-t(\frac{1}{2}\pi-|\mathrm{arc}\,y|)}t^{\frac{1}{2}+\epsilon}\,dt. \tag{24}$$

Putting $\gamma=\frac{1}{2}\pi-|\mathrm{arc}\,y|$, then*

$$0<\gamma^{-1}<n|y|,$$

so that the right member of (24) becomes

$$<Ak^{\frac{1}{4}+\epsilon}|y|^{-\frac{1}{2}-\epsilon}\gamma^{-\frac{1}{2}-\epsilon}\int\limits_0^\infty e^{-u}\,u\,du$$

$$<Ak^{\frac{1}{4}+\epsilon}|y|^{\frac{3}{4}+\epsilon}n^{\frac{3}{4}+\epsilon}.$$

Substituting into (23) and making use of†

$$|B(\chi)|<Ak^{\frac{1}{2}+\epsilon},$$

we get finally

$$\left|\int\limits_{(\frac{1}{2}+\epsilon)} y^{-s}\Gamma(s)\sum\frac{\alpha(m)\rho^m}{m^s}\,ds\right|<A(k|y|)^{\frac{3}{4}+\epsilon}n^{\frac{3}{4}+\epsilon}$$

$$<An^{\frac{3}{4}+\epsilon},$$

thus proving (7).

4. The principal result

Starting with (7) we prove

$$\left|R_\nu(n)-\frac{b^\nu n^{\nu-1}}{(\nu-1)!}\bar{S}(n)\right|<An^{\nu-\frac{5}{4}+\epsilon}, \tag{25}$$

where
$$\bar{S}(n)=\sum_{k=1}^\infty\frac{\tau^\nu(k)}{k^\nu}\sum_{\rho(k)}\rho^{-n}.$$

Since the proof is almost exactly like that of Theorem 249 of *Vorlesungen*, i, it is unnecessary to reproduce it here.

It remains, then, to identify $\bar{S}(n)$ with $S(n)$. To begin with, it is easily seen that, if we put

$$D(k)=\frac{\tau^\nu(k)}{k^\nu}\sum_{\rho(k)}\rho^{-n},$$

$$D(k_1k_2)=D(k_1)D(k_2)\text{ for }(k_1,k_2)=1.$$

* Cf. *Vorlesungen*, i, p. 213. † *Vorlesungen*, i, Th. 222.

This enables us to write

$$\bar{S}(n) = \prod_p \{1+D(p)+D(p^2)+\ldots\}. \tag{26}$$

Now*

$$\sum_{\rho(p^e)} \rho^{-n} = \begin{cases} -1 & \text{for } e=1 \\ 0 & \text{for } e>1 \end{cases} \quad (p \nmid n);$$

$$= \begin{cases} p^e - p^{e-1} & \text{for } e \leqslant f \\ -p^{e-1} & \text{for } e = f+1 \\ 0 & \text{for } e > f+1 \end{cases} \quad (p^f | n, p^{f+1} \nmid n).$$

Substituting into (26), we see immediately that

$$\bar{S}(n) = S(n);$$

this completes the proof of the

THEOREM. *For* $\nu \geqslant 3$; $R_\nu(n)$, $S(n)$, b *as defined in the Introduction; A depending only on* ϵ, $\epsilon > 0$,

$$\left| R_\nu(n) - \frac{b^\nu n^{\nu-1}}{(\nu-1)!} S(n) \right| < A n^{\nu-\frac{3}{2}+\epsilon}. \tag{2}$$

5. An application to the representation of a large integer as the sum of ν 'quadratfreie' numbers†

We suppose that

$$\alpha(n) = \mu^2(n).$$

Then by (4)

$$\tau(p) = \frac{-\frac{1}{p}}{1-\frac{1}{p^2}} = \frac{-p}{p^2-1}, \qquad \tau(p^2) = \frac{-1}{1-\frac{1}{p^2}} = \frac{-p^2}{p^2-1},$$

$$\tau(p^e) = 0 \text{ for } e > 2; \qquad b = \prod \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2}.$$

$S(n)$ can now be easily evaluated:

$$\prod_p \left(1 - \frac{\tau^\nu(p)}{p^\nu}\right) = \prod \left(1 - \frac{(-1)^\nu}{(p^2-1)^\nu}\right);$$

$$\prod_{p|n,\, p^2 \nmid n} = \prod \left(1 - \frac{(-1)^\nu}{(p^2-1)^\nu}\right)\left(1 - \frac{(-1)^\nu}{(p^2-1)^\nu}\right)^{-1} = 1;$$

* *Vorlesungen*, i, Th. 220.

† Cf. Linfoot and Evelyn, 'On a Problem in the Additive Theory of Numbers': *Math. Zeitschrift*, 30 (1929), 433–48, for a treatment of this problem along the lines of the Hardy-Littlewood solution of the Waring problem.

$$\prod_{p^2|n} = \prod \left(1+\frac{(-1)^\nu}{(p^2-1)^{\nu-1}}\right)\left(1-\frac{(-1)^\nu}{(p^2-1)^\nu}\right)^{-1};$$

$$S(n) = \prod_p \left(1-\frac{(-1)^\nu}{(p^2-1)^\nu}\right) \prod_{p^2|n} \frac{1+\dfrac{(-1)^\nu}{(p^2-1)^{\nu-1}}}{1-\dfrac{(-1)^\nu}{(p^2-1)^\nu}}.$$

This expression is obviously never zero. It is interesting to notice that the finite product depends only on primes appearing to at least the second power in n.

6. A second application

We shall here take

$$\alpha(n) = \frac{\sigma_r(n)}{n^r}, \qquad r \geqslant \tfrac{1}{2},$$

$\sigma_r(n)$ denoting the sum of the rth powers of the divisors of n. Then by (4)

$$\tau(p^e) = \frac{p^{-er}+p^{-(e+1)r+1}+p^{-(e+2)r+2}+\dots}{1+p^{-r-1}+p^{-2r-2}+\dots}$$
$$= p^{-er}.$$

Substituting into (3),

$$S(n) = \prod_p \left(1-\frac{1}{p^{(r+1)\nu}}\right) \prod_{p|n} \left(1+\frac{p}{p^{(r+1)\nu}}+\dots+\frac{p^f}{p^{f(r+1)\nu}}\right)$$
$$= \frac{\sigma_{(r+1)\nu-1}(n)}{n^{(r+1)\nu-1}} \frac{1}{\zeta\{(r+1)\nu\}}.$$

Since

$$b = \prod_p \left(1+\frac{1}{p^{r+1}}+\frac{1}{p^{2r+2}}+\dots\right)$$
$$= \prod \left(1-\frac{1}{p^{r+1}}\right)^{-1} = \zeta(r+1),$$

we find finally that

$$(R_\nu n) \sim \frac{\zeta^\nu(r+1)}{\zeta\{(r+1)\nu\}} \frac{n^{\nu-1}}{(\nu-1)!} \frac{\sigma_{(r+1)\nu-1}(n)}{n^{(r+1)\nu-1}}.$$

THE SUMMABILITY OF A FOURIER SERIES BY LOGARITHMIC MEANS

By G. H. HARDY

[Received 18 February 1931]

1. THERE is a simple necessary and sufficient condition for the Cesàro summability of the Fourier series of a bounded function. Suppose as usual that $f(t)$ is integrable in $(-\pi, \pi)$ and that

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos nt + b_n \sin nt). \tag{1.1}$$

Further, suppose that $f(t)$ is bounded near $t = x$, and that the question is that of the summability of the series, for $t = x$, to sum s. Then the condition

$$\int_0^t \phi(u)\,du = o(t) \tag{1.2}$$

or

$$\phi(t) = o(1) \qquad (C, 1), \tag{1.21}$$

where $t > 0$ and $\quad \phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\}$,

is necessary for summability by Cesàro means of any order, and sufficient for summability by Cesàro means of every positive order.*

There are simple bounded functions which do not satisfy this condition. Suppose, for example, that $x = 0$ and

$$f(t) = \cos(a \log |t|), \tag{1.3}$$

where $a > 0$. Then $b_n = 0$, while a_n behaves like

$$\frac{A\cos(a\log n) + B\sin(a\log n)}{n},$$

where A and B are constants, and $\sum a_n$ is not summable (C). The condition (1.2) is naturally not satisfied; all means of $f(t)$, round $t = 0$, oscillate, substantially like $f(t)$ itself.

The series $\sum a_n$ is, however, summable by Riesz's logarithmic means;† if

$$s_n = \tfrac{1}{2}a_0 + a_1 + a_2 + \ldots + a_n, \tag{1.4}$$

then

$$\frac{S_n}{\log n} = \frac{1}{\log n}\left(s_1 + \frac{s_2}{2} + \ldots + \frac{s_n}{n}\right) \tag{1.5}$$

tends to a limit s when $n \to \infty$, in which circumstances we write

* Hardy and Littlewood (**3**: see also **4**).

† For the general properties of Riesz's 'typical means' see the tract by Hardy and Riesz, and various papers by Zygmund in the *Math. Zeitschrift* and the *Bulletin de l'Acad. Polonaise*.

$$\tfrac{1}{2}a_0 + \sum a_n = s \qquad (R, 1). \tag{1.6}$$

These facts suggest that we should try to find some general theorem, of the type of that just quoted, concerning the Rieszian summability of the Fourier series of a bounded function. Actually Theorem A below covers a rather wider class of functions.

Some theorems concerning the Rieszian summability of trigonometrical series are known already. In particular Zygmund* has shown that the Fourier series of any integrable function, bounded or not, is summable $(R, 1)$ whenever (1.2) is satisfied; and he and Jacob† have investigated more general theorems of the same type. But none of these theorems covers such a function as (1.3), or is quite of the type desired.‡

2. THEOREM A. *Suppose that*

$$\Phi(t) = \int_0^t |\phi(u)|\, du = o\left(t \log \frac{1}{t}\right) \tag{2.1}$$

(a condition satisfied whenever $\phi(t) = o\left(\log \frac{1}{t}\right)$ *and in particular when* $f(t)$ *is bounded near* $t = x$*). Then a necessary and sufficient condition that the series should be summable* $(R, 1)$*, for* $t = x$*, to sum* s*, is that*

$$\psi(t) = \int_t^\pi \frac{\phi(u)}{u}\, du = o\left(\log \frac{1}{t}\right) \tag{2.2}$$

when $t \to 0$.

It is easily verified that (2.2) is satisfied whenever (1.2) is satisfied, and that the function (1.3) satisfies (2.2) but not (1.2). In proving the theorem we may make the usual simplifications, supposing that $f(t)$ is even and that $a_0 = 0$, $x = 0$, $s = 0$, so that $\phi(t) = f(t)$.

* Zygmund (**7**).

† Zygmund (**8**), Jacob (**5**).

‡ Though in other respects they are more general, since they apply to series which are derived series of Fourier series but not Fourier series themselves.

It may be useful to recall that (1.2) is not, when $f(t)$ is otherwise unrestricted, a sufficient condition for summability $(C, 1)$, though it is sufficient for summability $(C, 1+\delta)$, for any positive δ, or (in virtue of Zygmund's theorem) for summability $(R, 1)$. A series summable $(C, 1)$ is necessarily summable $(R, 1)$, but a series summable $(C, 1+\delta)$, for every positive δ, is not, so that Zygmund's theorem is not a corollary of the known results concerning Cesàro summability.

(i) *Proof that the condition is sufficient.* It is known* that if (as here) $f(x)$ is even and integrable, and $a_0 = 0$, then

$$\chi(x) = \int_x^\pi f(u)\,\tfrac{1}{2}\cot\tfrac{1}{2}u\,du \tag{2.3}$$

is also even and integrable, and

$$\chi(x) \sim \tfrac{1}{2}b_0 + \sum b_n \cos nx, \tag{2.31}$$

where
$$b_n = \frac{a_1+a_2+\ldots+a_{n-1}+\frac{1}{2}a_n}{n} \tag{2.32}$$

for $n > 0$. It is plain in the present case, from (2.2), that

$$\chi(x) = o\left(\log\frac{1}{x}\right) \tag{2.33}$$

for small positive x.

We write

$$s_n = a_1+a_2+\ldots+a_n, \qquad \sigma_n = s_1+s_2+\ldots+s_{n-1}, \tag{2.41}$$

$$t_0 = \tfrac{1}{2}b_0, \quad t_n = \tfrac{1}{2}b_0+b_1+\ldots+b_n \ (n>0), \quad \tau_n = t_0+t_1+\ldots+t_{n-1}. \tag{2.42}$$

Then

$$\sigma_n = \frac{1}{\pi}\int_0^\pi f(x)\frac{\sin^2\frac{1}{2}nx}{\sin^2\frac{1}{2}x}\,dx, \qquad \tau_n = \frac{1}{\pi}\int_0^\pi \chi(x)\frac{\sin^2\frac{1}{2}nx}{\sin^2\frac{1}{2}x}\,dx. \tag{2.43}$$

From (2.32) and (1.5)

$$\frac{s_n}{n} = b_n + o\left(\frac{1}{n}\right), \qquad S_n = t_n + o(\log n). \tag{2.5}$$

From (2.33) and (2.43),

$$\tau_n = \frac{1}{\pi}\int_0^{1/n} o\left(\log\frac{1}{x}\right)n^2\,dx + \frac{1}{\pi}\int_{1/n}^\pi o\left(\log\frac{1}{x}\right)\frac{dx}{x^2}$$

$$= o\left(n^2.\frac{1}{n}\log n\right) + o(n\log n) = o(n\log n).$$

Combining this with (2.42) and (2.5), we obtain

$$S_1+S_2+\ldots+S_n = \tau_n + o(n\log n) = o(n\log n). \tag{2.6}$$

But
$$(n+1)S_n - S_1 - S_2\ldots - S_n = s_1+s_2+\ldots+s_n,$$

and so, by (2.6),

$$S_n = \frac{S_1+S_2+\ldots+S_n}{n+1} + \frac{\sigma_n}{n+1} = \frac{\sigma_n}{n+1} + o(\log n). \tag{2.7}$$

* Hardy (**2**).

On the other hand

$$\frac{\sigma_n}{n} = \frac{4}{n\pi}\int_0^\pi f(x)\frac{\sin^2\frac{1}{2}nx}{x^2}\,dx + o(1) = J + o(1),$$

say, and

$$|J| \leqslant \frac{4}{n\pi}\int_0^{1/n} |f(x)|\frac{\sin^2\frac{1}{2}nx}{x^2}\,dx + \frac{4}{n\pi}\int_{1/n}^\pi \frac{|f(x)|}{x^2}\,dx = J_1 + J_2,$$

say. Here

$$J_1 \leqslant \frac{n}{\pi}\int_0^{1/n} |f(x)|\,dx = \frac{n}{\pi}\Phi\left(\frac{1}{n}\right) = o(\log n),$$

by (2.1), and

$$J_2 = \frac{4}{n\pi}\int_{1/n}^\pi \frac{\Phi'(x)}{x^2}\,dx < \frac{8}{n\pi}\int_{1/n}^\pi \frac{\Phi(x)}{x^3}\,dx + o(1)$$

$$= \frac{1}{n}\int_{1/n}^\pi \frac{o\left(\log\frac{1}{x}\right)}{x^2}\,dx + o(1) = o(\log n).$$

Collecting our results, we see that

$$\frac{\sigma_n}{n} = o(\log n),$$

and so, from (2.7), $S_n = o(\log n)$, the result required.

3. (ii) *Proof that the condition is necessary.* We now assume that $S_n = o(\log n)$ and deduce (2.2). We begin by showing, without using (2.1), that

$$\chi(x) = o\left(\log\frac{1}{x}\right) \qquad (C, 2), \tag{3.1}$$

i.e. that

$$\chi_2(x) = \int_0^x du \int_0^u \chi(v)\,dv = o\left(x^2 \log\frac{1}{x}\right). \tag{3.11}$$

The proof is much like that of Riemann's classical theorem. We have

$$2\frac{\chi_2(x)}{x^2} = \frac{1}{2}b_0 + \sum_1^\infty b_n\left(\frac{\sin\frac{1}{2}nx}{\frac{1}{2}nx}\right)^2$$

$$= \sum_0^\infty t_n\{g(nx) - g[(n+1)x]\},$$

where

$$g(u) = \left(\frac{\sin u}{u}\right)^2;$$

and so

$$2\frac{|\chi_2(x)|}{x^2} \leqslant \sum_0^\infty |t_n| \int_{nx}^{(n+1)x} |g'(u)|\, du = \sum_0^\infty |t_n| I_n, \tag{3.2}$$

say. But $g'(u)$ is $O(u)$ for small u, and $O(u^{-2})$ for large u; and so

$$I_n = O(nx^2)\ (nx \leqslant 1), \qquad I_n = O\left(\frac{1}{n^2 x}\right)\ (nx > 1).$$

Substituting in (3.2) and observing that, after (2.5), $t_n = o(\log n)$, we obtain

$$\frac{\chi_2(x)}{x^2} = \sum_{nx \leqslant 1} o(\log n) O(nx^2) + \sum_{nx>1} o(\log n) O\left(\frac{1}{n^2 x}\right)$$

$$= o\left(x^2 \sum_{nx\leqslant 1} n \log n\right) + o\left(\frac{1}{x} \sum_{nx>1} \frac{\log n}{n^2}\right) = o\left(\log\frac{1}{x}\right),$$

which is (3.1).

It follows that
$$\psi_2(x) = o\left(x^2 \log\frac{1}{x}\right) \tag{3.3}$$

(suffixes denoting, as before, integrations from 0 to x). On the other hand it is easily verified* that

$$\tfrac{1}{2}x^2\psi(x) = \psi_2(x) - \tfrac{1}{2}xf_1(x) - \tfrac{1}{2}f_2(x). \tag{3.4}$$

The last two terms are $o\left(x^2 \log\frac{1}{x}\right)$, by (2.1); and it therefore follows from (3.3) that

$$\psi(x) = o\left(\log\frac{1}{x}\right),$$

which is (2.2). This completes the proof of the theorem.

4. We might naturally express (2.2) by saying that $f(t)$ is 'continuous $(R, 1)$' for $t = x$; Theorem A then asserts that, if (2.1) is satisfied (and in particular if f is bounded near $t = x$), *continuity* $(R, 1)$ *is a necessary and sufficient condition for summability* $(R, 1)$.

It is natural to ask what happens when the condition (2.1) is dropped, and the answer suggested by the analogy with Cesàro summability is as follows. We must begin by defining summability (R, k) and continuity (R, l). We may then expect a double scale of theorems like those investigated first by Littlewood and myself†, and then, more precisely, by Bosanquet and Paley.‡ None of these theorems, however, will be of the 'necessary and sufficient' type; for

* For example by differentiation. † Hardy and Littlewood (**3**).

‡ Bosanquet (**1**), Paley (**6**).

that, it will be essential to close the cycle by some such condition as (2.1). It might be worth while to push the analysis a little farther (though hardly to develop it in full detail), but I confine myself to what seems to be the simplest and most interesting case.

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THE ZEROS OF CERTAIN INTEGRAL FUNCTIONS. (II)

By MARY L. CARTWRIGHT

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1.1. In a previous paper† I considered functions of the form

$$f(z) = f(x+iy) = f(re^{i\theta}) = \int_{-1}^{1} e^{zt}\phi(t)\,dt, \tag{1.11}$$

where $\phi(t)$ is a complex function, integrable in the sense of Lebesgue, and $\phi(t)$ tends to a finite limit other than 0 at each end. The object of this paper is to consider the cases in which $\phi(t)$ tends to 0 or ∞ at one or both ends. If $\phi(t) \to \infty$ as $t \to \pm 1$, we need lighter restrictions in order to obtain equivalent results, and if $\phi(t) \to 0$, heavier ones, as the following example shows.

Suppose that

$$\begin{aligned} \phi(t) &= 1-t \qquad (0 < t \leqslant 1) \\ &= 1+t \qquad (-1 \leqslant t \leqslant 0), \end{aligned}$$

then

$$\phi(t) = \int_{-1}^{t} \phi'(u)\,du, \qquad f(z) = \frac{4\sinh^2(\frac{1}{2}z)}{z^2}.$$

Hence, although $\phi(t)$ is an integral, the zeros of $f(z)$ occur in pairs, and not as in Theorem VI of Z.

In the above example all the zeros still lie on the imaginary axis, but Titchmarsh‡ has constructed a function of the form (1.11) which has an infinity of real zeros.

1.2. I suppose first that

$$\begin{aligned} \phi(t) &\sim C(1-t)^q \qquad \text{as } t \to 1, \\ \phi(t) &\sim C'(1+t)^q \qquad \text{as } t \to -1, \end{aligned}$$

where $q > -1$, $C \neq 0$, $C' \neq 0$. We may suppose without loss of generality that $C = C' = 2^q$, so that we may write

$$\phi(t) \sim (1-t^2)^q \tag{1.21}$$

as $t \to \pm 1$. The other cases in which $C \neq 0$, $C' \neq 0$ may be reduced to this by simple transformations as in Z, § 1.4.

Theorem I gives asymptotic values for $f(z)$ in the regions

$$|\cos\theta| > \delta > 0, \qquad r > r_0(\delta),$$

† M. L. Cartwright, *Quart. J. of Math.* (Oxford), 1 (1930), 38–59. I shall refer to this paper in future as Z.

‡ E. C. Titchmarsh, *Proc. London Math. Soc.* (2) 25 (1926), 291, Theorem III.

and shows that $f(z)$ has no zeros in those regions. This is the only theorem deduced from (1.21) alone. We seem unable to improve on the formula

$$n(\rho) \sim 2\rho/\pi,$$

which was proved by Titchmarsh† for any function of the form (1.11), without further conditions on $\phi(t)$.

When q is an integer, we impose the conditions used in Z on the qth differential coefficient of $\phi(t)$, instead of on $\phi(t)$ itself. If q is fractional we have a complication arising from the irregularity of the dominant term itself; for if $q = m+\alpha-1$, where m is 0 or an integer $(0 < \alpha < 1)$ then the mth differential coefficient of $\phi(t)$ becomes infinite at $t = \pm 1$. Theorems V and VII give typical simple cases for $q = \alpha-1$; the cases in which $m > 0$ reduce to this on integrating m times. Theorems VIII and IX are the most general results of their kind; they include Theorems II, III, V, and VII. Generalizations of IV are also indicated briefly; I and VI stand by themselves. The use of integrated Lipschitz conditions gives very great generality, and is equally suitable whether q is an integer or not. By separating the term $(1-t^2)^q$ from the rest we avoid confusing the well-defined irregularity of $(1-t)^{\alpha-1}$, where $0 < \alpha < 1$, with the general irregularity of $\phi(t)$.

In §§ 7.1—7.3 I use the methods evolved in the previous theorems to prove a new type of theorem. Returning to the case $q = 0$, considered in Z, I suppose that $\phi(t)$ belongs to $\mathrm{Lip}(\alpha, p)$, where $\alpha p > 1$. The error term for $n(\rho)$ in these results is $O(r^{(1-\alpha)p'})$, where

$$p' = p/(p-1),$$

while a_n, the nth Fourier coefficient of $\phi(t)$, is of the form $O(n^{-\alpha})$. Hence the analogue observed in Z‡ fails here. Some results can be deduced directly from the Fourier coefficients, but only if a_n is $o(n^{-1})$ or even smaller; these, however, I must leave for another time.

In all these cases the zeros are distributed more or less like those of $z^{-q-\frac{1}{2}}J_{q+\frac{1}{2}}(-iz)$, i.e. like those of $z^{-q-1}\sinh(z\pm\frac{1}{2}q\pi i)$ according as $y \gtrless 0$.

These methods can also be applied when

$$\phi(t) \sim C(1-t)^q \qquad \text{as } t \to 1,$$
$$\phi(t) \sim C'(1-t)^{q'} \qquad \text{as } t \to -1,$$

† E. C. Titchmarsh, loc. cit., Theorem IV. ‡ See § 3.1.

where $q \neq q'$. We reduce as before to the form

$$\phi(t) \sim (1-t)^{q}(1+t)^{q'}. \tag{1.22}$$

The zeros of the typical function are given approximately by

$$\left.\begin{aligned} x &= \tfrac{1}{2}(q-q')\log 2n\pi + \tfrac{1}{2}\log\frac{\Gamma(q'+1)}{\Gamma(q+1)} + \epsilon_n, \\ y &= \pm[n+\tfrac{1}{4}(q+q')]\pi + \epsilon_n, \end{aligned}\right\} \tag{1.23}$$

where ϵ_n is a number (not necessarily the same in both places) which tends to 0 as $n \to \infty$. When $\phi(t)$ is of the form (1.22), the zeros of $f(z)$ approximate to (1.23); and owing to the lop-sided arrangement of the zeros with regard to the imaginary axis we have to impose heavier conditions on $\phi(t)$ at one end than are needed at the other. This makes it difficult to state satisfactory theorems. I shall therefore only give a brief sketch of the chief points of difference between these functions and the previous ones.

Finally, I discuss briefly functions for which $\phi(t)$ tends to 0 exponentially at the ends of the interval of integration.

1.3. I use the notation of Z as far as possible, that is to say: $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}, \ldots$, where $0 < r_1 \leqslant r_2 \leqslant \ldots$, are the zeros of $f(z)$; $n(\rho)$ is the number of zeros for which $|z| \leqslant \rho$; $N(\rho) = \int\limits_0^\rho \frac{n(u)}{u}\,du$; a_n is the nth Fourier coefficient of $\phi(t)$; $[\rho]$ denotes the integral part of ρ.

I am indebted to Professor Hardy for many valuable suggestions particularly with regard to §§ 4.1—4.4 and § 9.1.

2.1. Theorem I. *If $\phi(t)$ satisfies (1.21), then, given any $\epsilon > 0, \delta > 0$, we can choose $r_0 = r_0(\delta, \epsilon)$ so that*

$$\begin{aligned} \frac{z^{q+1}f(z)}{2^q\Gamma(q+1)} &= e^{-z-q\pi i}(1+\zeta) \qquad & (-\tfrac{3}{2}\pi+\delta \leqslant \theta \leqslant -\tfrac{1}{2}\pi-\delta) \\ &= e^{z}(1+\zeta) & (-\tfrac{1}{2}\pi+\delta \leqslant \theta \leqslant \tfrac{1}{2}\pi-\delta) \\ &= e^{-z+q\pi i}(1+\zeta) & (\tfrac{1}{2}\pi+\delta \leqslant \theta \leqslant \tfrac{3}{2}\pi-\delta), \end{aligned}$$

where ζ is not necessarily the same in each case and $|\zeta| < \epsilon$ for every $r > r_0$.

Corollary. *If $\phi(t)$ satisfies (1.21), then $f(z)$ has no zeros in the regions $|\theta| \leqslant \frac{1}{2}\pi-\delta$, $|\pi-\theta| \leqslant \frac{1}{2}\pi-\delta$, $r > r_0(\delta)$.*

It is convenient (but not necessary) to use these asymptotic expressions to prove some of the later theorems. The apparent difference

between the first and third is only due to the many-valued factor z^q on the left-hand side.†

Proof. Suppose first that $x > 0$. We may write $\phi(t) = (1-t^2)^q\psi(t)$, where $\psi(t)$ is continuous and equal to 1 at $t = 1$, and, given any $\epsilon_1 > 0$, we can choose η so that

$$|\psi_1(t)| = |2^q-(1+t)^q\psi(t)| < \epsilon_1 \tag{2.11}$$

for $1-\eta < t \leqslant 1$. Hence

$$\begin{aligned} f(z) &= 2^q \int_{1-\eta}^{1} e^{zt}(1-t)^q\,dt + \int_{1-\eta}^{1} e^{zt}(1-t)^q\,\psi_1(t)\,dt + \int_{-1}^{1-\eta} e^{zt}\phi(t)\,dt \\ &= I_1+I_2+I_3. \end{aligned}$$

It is easy to see that

$$|I_3| < e^{(1-\eta)x} \int_{-1}^{h\eta} |\phi(t)|\,dt < Ae^{(1-\eta)x},‡$$

while
$$\begin{aligned} |I_2| &< \epsilon_1 \int_{1-\eta}^{1} e^{xt}(1-t)^q\,dt = \epsilon_1 e^x \int_0^\eta e^{-xt}t^q\,dt \\ &< \epsilon_1 e^x \int_0^\infty e^{-xt}t^q\,dt = \epsilon_1 \frac{e^x\Gamma(q+1)}{x^{q+1}}. \end{aligned} \tag{2.12}$$

Also
$$\begin{aligned} I_1 &= 2^q e^z \int_0^\eta e^{-zt}t^q\,dt = 2^q e^z\left\{\int_0^\infty - \int_\eta^\infty\right\}e^{-zt}t^q\,dt \\ &= 2^q e^z\Gamma(q+1)z^{-q-1}+O\{e^{(1-\eta)x}\}. \end{aligned}$$

Putting these results together, we have

$$z^{q+1}f(z) = \Gamma(q+1)2^q e^z(1+\zeta),$$

where $|\zeta| < \epsilon_1(\cos\theta)^{-q-1}+Ar^{q+1}e^{-\eta x} < \epsilon$, provided that $|\theta| \leqslant \frac{1}{2}\pi-\delta$, $r > r_0(\epsilon,\delta)$. For we can first fix δ, and then choose ϵ_1 so that $\epsilon_1(\sin\delta)^{-q-1} < \frac{1}{2}\epsilon$. We then choose η so that (2.11) is satisfied, and lastly r_0 so that $Ar_0{}^{q+1}e^{-\eta r_0\sin\delta} < \frac{1}{2}\epsilon$.

The other two asymptotic values are obtained in a similar fashion by dividing the range of integration near $t = -1$.

3.1. Suppose that $q = m$, where m is a positive integer. In the first example of § 1.1, $\phi(t) \sim 1\mp t$ as $t \to \pm 1$, and $\phi'(t)$ is of bounded variation, while the zeros are like those of the example in Z, § 3.11, in which $\phi(t) \to 1$ at the ends and $\phi(t)$ itself is of bounded variation.

† Compare the formulae for Bessel functions, see G. N. Watson, *Theory of Bessel Functions*, Cambridge (1922), 48.

‡ Throughout the paper I use A to denote a positive constant, not necessarily the same in different places.

This suggests that we must now impose conditions on $\phi^{(m)}(t)$ instead of $\phi(t)$. When this is done, we obtain results similar to those of Z, and the example shows that we cannot expect anything much better.

The continuity condition at the ends corresponding to that used in Z is that $\phi^{(m)}(t)$ is continuous. Suppose that this is satisfied; it then follows from Taylor's theorem that

$$\phi(t) = \phi(1)+(t-1)\phi'(1)+...+\frac{(t-1)^m}{m!}\phi^{(m)}\{1+\xi(t-1)\}, \quad (3.11)$$

where $0 \leqslant \xi < 1$, near $t = 1$. Comparing (1.21) and (3.11) we have

$$\phi(1) = \phi'(1) = ... = \phi^{(m-1)}(1) = 0, \quad (3.12)$$

and $\phi^{(m)}(1) = m!(-2)^m$. Similarly

$$\phi(-1) = \phi'(-1) = ... = \phi^{(m-1)}(-1) = 0, \quad (3.13)$$

and $\phi^{(m)}(-1) = m!2^m$.

3.2. THEOREM II. *Suppose that $\phi(t)$ is of the form* (1.21), *and that $\phi^{(m)}(t)$ is an integral. Then the zeros of $f(z)$ are determined asymptotically by the formula*

$$z = \pm(n+\tfrac{1}{2}m)\pi i+\epsilon_n, \quad (3.21)$$

where n is a positive integer and $\epsilon_n \to 0$; also

$$n(\rho) = 2\left[\frac{\rho}{\pi}-\tfrac{1}{2}m\right], \quad (3.22)$$

provided that ρ is large and that $\rho/\pi-\frac{1}{2}m$ is not too near an integer.

We can integrate $m+1$ times, and we have

$$\begin{aligned} f(z) &= \frac{(-1)^m}{z^m}\int_{-1}^{1} e^{zt}\phi^{(m)}(t)\,dt \\ &= \frac{(-1)^m}{z^{m+1}}\{e^z\phi^{(m)}(1)-e^{-z}\phi^{(m)}(-1)\}-\frac{(-1)^m}{z^{m+1}}\int_{-1}^{1} e^{zt}\phi^{(m+1)}(t)\,dt \\ &= \frac{\Gamma(m+1)2^m}{z^{m+1}}\{e^z-(-1)^m e^{-z}\}-I. \end{aligned}$$

But, by Titchmarsh's lemma,† $I = o\{e^{|x|}/r^{m+1}\}$, so that

$$\frac{z^{m+1}f(z)}{\Gamma(m+1)2^m} = e^z(1+\epsilon_r)-(-1)^m e^{-z}(1+\epsilon_r),$$

where ϵ_r denotes generally a function of r and θ which tends to 0 uniformly in θ when $r \to \infty$.

At a zero $$e^{2z} = e^{\pm m\pi i}(1+\epsilon_r),$$

so that $$2z = \pm(2n+m)\pi i+\epsilon_n.$$

† E. C. Titchmarsh, loc. cit., Lemma 2.2, or Z, Lemma A.

We can complete the proof of (3.21) as in Z by showing that one and only zero is associated with each large integral value of n. We prove (3.22) by showing that $z^{m+1}f(z)$ has the same number of zeros as $\sinh(z-\frac{1}{2}m\pi i)$ inside $|z|=\rho$, provided that the circle does not pass too near a zero of $\sinh(z-\frac{1}{2}m\pi i)$.

3.3. THEOREM III. *Suppose that* $\phi(t)$ *is of the form* (1.21), *and that* $\phi^{(m)}(t)$ *is of bounded variation and continuous at* $t=\pm 1$. *Then*

(i) *all the zeros of* $f(z)$ *lie in the strip* $|x|<K$ *for some* K;

(ii) $n(\rho)=\dfrac{2\rho}{\pi}+O(1)$;

(iii) $N(\rho)=\dfrac{2\rho}{\pi}-(m+1)\log\rho-\log|f(0)|+o(1)$.

We integrate m times and then apply the methods of Z, §§ 3.12—3.42.† The results follow in precisely the same way.

3.4. Similarly we have

THEOREM IV. *Suppose that* $\phi(t)$ *is of the form* (1.21), *and that* $\phi^{(m)}(t)$ *is continuous with modulus of continuity* ω.‡ *Then*

(i) *all the zeros lie in the region* $|x|<Kr\omega(r^{-1})$,

(ii) $n(\rho)=\dfrac{2\rho}{\pi}+O\{\rho\omega(\rho^{-1})\}$,

(iii) $$N(\rho)=\frac{2\rho}{\pi}-(m+1)\log\rho-\log|f(0)|+O\{\rho\omega^2(\rho^{-1})\}+ +O\left\{\omega(\rho^{-1})\log\frac{1}{\omega(\rho^{-1})}\right\}.$$

4.1. Now suppose that

$$\phi(t)\sim(1-t^2)^{\alpha-1}, \tag{4.11}$$

where $0<\alpha<1$, as $t\to\pm 1$, and consider first the case corresponding to Theorem II. A simple transformation shows that $\frac{1}{2}f(n\pi i)$ is actually the complex Fourier coefficient of $\phi(t/\pi)$. Bromwich,§ Young,|| and Haslam-Jones** have obtained asymptotic values for the Fourier coefficients of different classes of functions satisfying

† On p. 43 of Z in the last line, read $\int_{1-\delta}^{1}|d\phi|$ instead of $\max\limits_{1-\delta\leqslant t_1<t_2\leqslant 1}|\phi(t_2)-\phi(t_1)|$.

‡ Ch.-J. de la Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle*, or Z, § 4.1.

§ T. J. I'a Bromwich, *Theory of Infinite Series* (2nd ed.), 494.

|| W. H. Young, *Proc. Royal Soc.* (A), 93 (1917), 49–55.

** U. S. Haslam-Jones, *Journal London Math. Soc.* 2 (1927), 151–4; see the note added at the end.

(4.11). In each case the working holds equally well for non-integral n so that we can easily obtain from it an asymptotic value for $f(iy)$. From this we deduce the required result by a theorem of Phragmén and Lindelöf. This method is not directly applicable when $q=0$ or an integer; for we have to go through the working in order to find out the dominant terms. For example, if $q=0$ and $\phi(t)$ is an integral,

$$f(iy)=\frac{e^{iy}-e^{-iy}}{iy}+o\left(\frac{1}{|y|}\right);$$

although, since $e^{n\pi i}=e^{-n\pi i}$, we have $a_n=o(n^{-1})$. A mere order condition on $f(iy)$ is insufficient here; we must be able to separate the principal terms from the rest.

4.2. Using Bromwich's work, which is the simplest of the three, we have

THEOREM V. *Suppose that $\phi(t)=(1-t^2)^{\alpha-1}\psi(t)$, where $\psi(t)$ is of bounded variation in $(-1,-1+\eta)$ and $(1-\eta,1)$, and $\psi(t)\to 1$ as $t\to\pm 1$. Suppose further that $\phi(t)$ belongs to* Lip* α† *in $(-1+\eta, 1-\eta)$. Then the zeros of $f(z)$ are determined by*

$$z=\pm\{n+\tfrac{1}{2}(\alpha-1)\}\pi i+\epsilon_n, \tag{4.21}$$

where n is a positive integer and $\epsilon_n\to 0$.

We have

$$f(iy)=\left\{\int_{-1}^{-1+\eta}+\int_{-1+\eta}^{1-\eta}+\int_{1-\eta}^{1}\right\}e^{iyt}\phi(t)\,dt=I_1+I_2+I_3.$$

Since $(1+t)^{\alpha-1}\psi(t)$ is of bounded variation in $(1-\eta,1)$, it follows from Bromwich's work that

$$I_3=e^{iy}\int_0^{\eta}e^{-iyt}t^{\alpha-1}(2-t)^{\alpha-1}\psi(1-t)\,dt$$
$$=\Gamma(\alpha)e^{iy}2^{\alpha-1}(iy)^{-\alpha}+o(|y|^{-\alpha}).$$

Similarly $$I_1=\Gamma(\alpha)e^{-iy}2^{\alpha-1}(iy)^{-\alpha}e^{+\alpha\pi i}+o(|y|^{-\alpha}).$$

Also

$$I_2=\int_{-1+\eta}^{1-\eta}e^{iyt}\phi(t)\,dt=-\int_{-1+\eta+\pi/y}^{1-\eta+\pi/y}e^{iyt}\phi\left(t-\frac{\pi}{y}\right)dt$$
$$=\tfrac{1}{2}\int_{-1+\eta+\pi/y}^{1-\eta-\pi/y}e^{iyt}\left\{\phi(t)-\phi\left(t-\frac{\pi}{y}\right)\right\}dt+\tfrac{1}{2}\left\{\int_{-1+\eta}^{-1+\eta+\pi/y}+\int_{1-\eta-\pi/y}^{1-\eta}\right\}e^{iyt}\phi(t)\,dt.$$

† $\phi(t)$ belongs to Lip α in (a,b) if $|\phi(t+h)-\phi(t)|=O(h^{\alpha})$ $(a\leqslant t<t+h\leqslant b)$, and to Lip* α if $|\phi(t+h)-\phi(t)|=o(h^{\alpha})$.

Hence

$$|I_2| \leqslant \tfrac{1}{2}\int\limits_{-1+\eta}^{1-\eta}\left|\phi(t)-\phi\left(t-\frac{\pi}{y}\right)\right|dt+\tfrac{1}{2}\left\{\int\limits_{-1+\eta}^{-1+\eta+\pi/y}+\int\limits_{1-\eta-\pi/y}^{1-\eta}\right\}|\phi(t)|\,dt$$

$$= o(|y|^{-\alpha})+O(|y|^{-1}) = o(|y|^{-\alpha}).$$

We now have

$$F(z) = z^{\alpha}e^{-z}f(z)-\Gamma(\alpha)2^{\alpha-1}(1-e^{-2z+\alpha\pi i}) = o(1) \tag{4.22}$$

for $\theta = \frac{1}{2}\pi$, and also, by Theorem I, for $\theta = 0$. It is easy to see that $F(z) = O(r^{\alpha})$ uniformly for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, and it follows from a theorem of Phragmén and Lindelöf † that (4.22) holds uniformly for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Using a similar process for $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$, we have

$$f(z) = \frac{\Gamma(\alpha)2^{\alpha-1}}{z^{\alpha}}(e^{z}+e^{-z+\alpha\pi i})+o\left(\frac{e^{|x|}}{r^{\alpha}}\right)$$

for $y \geqslant 0$. Hence at a zero in the upper half-plane

$$e^{z}(1+\epsilon_r) = e^{-z+(\alpha-1)\pi i}(1+\epsilon_r)$$

so that $$z = \{n+\tfrac{1}{2}(\alpha-1)\}\pi i+\epsilon_n.$$

In the lower half-plane we have

$$f(z) = \frac{\Gamma(\alpha)2^{\alpha-1}}{z^{\alpha}}(e^{z}+e^{-z-\alpha\pi i})+o\left(\frac{e^{|x|}}{r^{\alpha}}\right),$$

so that $$z = -\{n+\tfrac{1}{2}(\alpha-1)\}\pi i+\epsilon_n.$$

4.3. In general we have

THEOREM VI. *Suppose that $\phi(t)$ satisfies* (4.11) *with $0 < \alpha \leqslant 1$, and that*

$$(iy)^{\alpha}f(iy) = \Gamma(\alpha)2^{\alpha-1}(e^{iy}+e^{-iy\pm\alpha\pi i})+o(1), \tag{4.31}$$

for $y \gtrless 0$. Then the zeros of $f(z)$ are given by (4.21).

It is easy to show that functions for which $\phi(t)$ satisfies Young's‡ or Haslam-Jones's§ conditions at $t = \pm 1$ satisfy (4.31), and therefore have their zeros given by (4.21). Young's condition at the point $t = a$ is that

$(t-a)^{2-\alpha}\phi'(t)$ is bounded in some interval $(a-\eta, a+\eta)$ and tends to 1 as $t \to a$. (4.32)

Haslam-Jones's condition is that

$\phi(t) = (t-a)^{\alpha-1}\psi(t)$, where $\psi(t) \to 1$ as $t \to a$, and $\psi(t)$ satisfies 'Young's convergence criterion',|| i.e.

$$\int\limits_{a}^{a+t}|d\{(u-a)\psi(u)\}| = O(t). \tag{4.33}$$

† G. Pólya and G. Szegö, *Aufgaben und Lehrsätze*, vol. i, Berlin (1925), 149.

‡ loc. cit. § loc. cit.

|| E. W. Hobson, *Theory of Functions of a Real Variable*, 3rd ed., 2, 532.

4.4. Corresponding to Theorem III we have

THEOREM VII. *Suppose that $\phi(t)$ satisfies the conditions of Theorem V except that it belongs to* Lip α *instead of* Lip* α *in* $(-1+\eta, 1-\eta)$. *Then* (i) *and* (ii) *of Theorem III are satisfied, and*

(iii) $$N(\rho) = \frac{2\rho}{\pi} - \alpha \log \rho + \log|f(0)| + o(1).$$

We write
$$f(z) = \left\{ \int_{1-\eta}^{1} + \int_{-1}^{-1+\eta} \right\} e^{zt}\phi(t)\,dt + \int_{-1+\eta}^{1-\eta} e^{zt}\phi(t)\,dt$$
$$= f_1(z) + f_2(z).$$

As in Theorem V, we have
$$f_1(z) = \frac{\Gamma(\alpha)2^{\alpha-1}}{z^\alpha}(e^z + e^{-z-\alpha\pi i}) + o\left(\frac{e^{|x|}}{r^\alpha}\right) \tag{4.41}$$
for $y \geqslant 0$. Also
$$f_2(x+iy) = \int_{-1+\eta}^{1-\eta} e^{(x+iy)t}\phi(t)\,dt = -\int_{-1+\eta+\pi/y}^{1-\eta+\pi/y} e^{(x+iy)t-x\pi/y}\phi\left(t-\frac{\pi}{y}\right)dt$$
$$= \tfrac{1}{2}\int_{-1+\eta+\pi/y}^{1-\eta} e^{(x+iy)t}\left\{\phi(t) - e^{-x\pi/y}\phi\left(t-\frac{\pi}{y}\right)\right\}dt +$$
$$+\tfrac{1}{2}\left\{\int_{-1+\eta}^{-1+\eta+\pi/y} + \int_{1-\eta-\pi/y}^{1-\eta}\right\} e^{zt}\phi(t)\,dt$$
$$= I_1 + I_2.$$

It is easy to see that $$|I_2| < Ae^{(1-\eta)x}/y. \tag{4.42}$$

Also
$$|I_1| \leqslant e^{(1-\eta)x}\int_{-1+\eta}^{1-\eta}\left|\phi(t) - \phi\left(t-\frac{\pi}{y}\right) + (1-e^{-x\pi/y})\phi\left(t-\frac{\pi}{y}\right)\right|dt \tag{4.43}$$
$$< e^{(1-\eta)x}\int_{-1+\eta}^{1-\eta}\left|\phi(t) - \phi\left(t-\frac{\pi}{y}\right)\right|dt + e^{(1-\eta)x}(1-e^{-x\pi/y})\int_{-1+\eta}^{1-\eta}|\phi(t)|\,dt$$
$$< A_1\frac{e^{(1-\eta)x}}{y^\alpha} + A_2 x\frac{e^{(1-\eta)x}}{y} < (A_1 + A_2 x)\frac{e^{(1-\eta)x}}{y^\alpha}$$
as $y \to \infty$. Hence we can choose K so that
$$|z^\alpha e^{-z} f_2(z)| < (A_3 + A_2 x)e^{-\eta x} < \tfrac{1}{2} \tag{4.44}$$
for $x = K$. It follows, as before, from the theorem of Phragmén and Lindelöf that (4.44) holds uniformly for $x \geqslant K$. We now have
$$z^\alpha f(z) = \Gamma(\alpha)2^{\alpha-1}e^z(1+\zeta),$$

where $|\zeta| < 1$ for $x \geqslant K$. Then (i) follows immediately, and it follows from (4.41), (4.42), and (4.43) that $f(z) = O(r^{-\alpha})$ for all fixed x. We can then apply the methods of Z to $z^{\alpha}f(z)$, and obtain (ii) and (iii).

4.5. If $\phi(t) = (1-t^2)^{\alpha-1}\psi(t)$, where $\psi(t)$ is continuous with modulus of continuity ω, we have results corresponding to those of Theorem IV. The proof is like those of Theorems X, XI, and XII of Z, except that we write $\phi(t) = (1-t)^{\alpha-1}2^{\alpha-1}+\psi_1(t)$ in the end interval.

4.6. Theorems V and VII are by no means the most general of their kind, and the abrupt change in the hypotheses at the points $t = \pm(1-\eta)$ is quite artificial. However they are included in Theorems VIII and IX. Theorem VI, although very general, is unsatisfactory without some general criterion as to whether the hypothesis is satisfied for any given ϕ.

5.1. Suppose that $\phi(t)$ is of the form (1.21), where now $q = m+\alpha-1$, m is zero or a positive integer, and $0 < \alpha \leqslant 1$. Suppose also that

$$\phi^{(m)}(t) = \frac{d^m}{dt^m}(1-t^2)^q + o(1-t^2)^{\alpha-1}, \tag{5.11}$$

as $t \to \pm 1$. This is equivalent to the continuity conditions used in §§ 3.1 and 4.1, and it follows that (3.12) and (3.13) still hold for $0 < \alpha < 1$.

Consider first the case corresponding to Theorems III and VII. Hardy and Littlewood† have shown that functions of bounded variation are equivalent to functions which belong to Lip (1, 1). It therefore seems natural, when $\alpha < 1$, to replace the hypothesis of bounded variation by '$\phi(t)$ belongs to Lip $(\alpha, 1)$'.† The definition of Lip $(\alpha, 1)$ is as follows:‡ *Suppose that $\phi(t)$ is integrable and periodic with period $b-a$, and suppose that*

$$\int_a^b |\phi(t+h)-\phi(t)|\, dt = O(|h|^{\alpha}), \tag{5.12}$$

where $0 < \alpha \leqslant 1$, as $h \to 0$; then $\phi(t)$ belongs to Lip $(\alpha, 1)$ *in* (a, b). We say that $\phi(t)$ *belongs to* Lip* $(\alpha, 1)$, *where* $0 < \alpha < 1$, *if the above definition is satisfied with o instead of O.* If it is satisfied for $\alpha = 1$ with o, then there is a constant c such that $\phi(t) = c$ almost everywhere.§ For Fourier coefficients, $\phi(t)$ an integral gives an o-result compared

† G. H. Hardy and J. E. Littlewood, *Math. Zeit.* 28 (1928), 619.

‡ G. H. Hardy and J. E. Littlewood, loc. cit., 612.

§ E. C. Titchmarsh, *Journal London Math. Soc.* 2 (1927), 36, 37.

with an O-result for $\phi(t)$ of bounded variation. For the purposes of this paper, therefore, I shall say that *$\phi(t)$ belongs to* Lip* $(1,1)$ *if it is an integral.* We now have perfectly smooth generalizations of the properties of bounded variation and of being an integral which are applicable for $0<\alpha\leqslant 1$. We say further that *$\phi(t)$ belongs to* Lip $(q,1)$, *where $q=m+\alpha-1$, if*

$$\phi^{(m-1)}(t)=\int_{t_0}^{t}\phi^{(m)}(t)\,dt, \tag{5.13}$$

and $\phi^{(m)}(t)$ belongs to Lip $(\alpha,1)$. We may of course replace Lip by Lip* in this definition.

The relations of the integrated Lipschitz conditions to those used in §§ 4.1—4.4 are given by the following lemmas.

LEMMA A. *$\phi(t)$ belongs to* Lip* $(\alpha,1)$, *where $\alpha<1$, in $(-1,1)$, if it is defined outside the interval by periodicity and satisfies one or other of the following conditions inside the interval:*

(1) *$\phi(t)=(t-a)^{\alpha-1}\psi(t)$ $(-1\leqslant a\leqslant 1)$, where $\psi(t)\to 0$ as $t\to a$, and $\psi(t)$ is of bounded variation.*†

(2) *$(t-a)^{2-\alpha}\phi'(t)$ is bounded and tends to 0 as $t\to a$.*‡

(3) *$\phi(t)=(t-a)^{\alpha-1}\psi(t)$, where $\psi(t)\to 0$ as $t\to a$, and $\psi(t)$ satisfies Young's convergence criterion.*§

(4) *$\phi(t)=(t-a)^{\beta-1}\psi(t)$, where $\beta>\alpha$, and $\psi(t)$ belongs to*

Lip* $(\alpha+1-\beta)$.

(5) *$\phi(t)$ belongs to* Lip* α.

The last class of functions obviously belongs to Lip* $(\alpha,1)$, and the proof for (4) is very straightforward. (3) includes (2) and (1), and the proof is a little more difficult.

LEMMA B. *$\phi(t)$ belongs to* Lip $(\alpha,1)$ *in $(-1,1)$, if it is defined outside the interval by periodicity, and satisfies one or other of the following conditions inside the interval:*

(1) *$\phi(t)$ satisfies* (1), (2), *or* (3) *of Lemma B except that $\psi(t)$ or $(t-a)^{2-\alpha}\phi'(t)$ may tend to a limit other than* 0.

(2) *$\phi(t)=(t-a)^{\beta-1}\psi(t)$, where $\beta>\alpha$, and $\psi(t)$ belongs to* Lip $(\alpha+1-\beta)$.

(3) *$\phi(t)$ belongs to* Lip α.

5.2. It is easy to verify that, when $\alpha<1$, $(1-t^2)^{\alpha-1}$ does not belong

† Compare Theorem V.

‡ Compare (4.32). If $\phi(t)$ satisfies (4.32), $\phi(t)-(t-a)^{\alpha-1}$ satisfies Lemma A (2).

§ Compare (4.33), and see footnote.

to $\text{Lip}^*(\alpha, 1)$, but only to $\text{Lip}(\alpha, 1)$. Hence, when $\phi^{(m)}(t)$ is of the form (5.11), it cannot, in general, belong to $\text{Lip}^*(\alpha, 1)$. We therefore subtract $(1-t^2)^q$ from $\phi(t)$ before imposing our hypotheses. This makes no difference if q is an integer.

THEOREM VIII. *Suppose that $\phi(t)$ is of the form* (1.21), *and satisfies* (5.11), *and that $\phi(t) = (1-t^2)^q + \chi(t)$, where $\chi(t)$ belongs to* $\text{Lip}^*(q+1, 1)$. *Then the zeros of $f(z)$ are given by*

$$z = \pm(n+\tfrac{1}{2}q)\pi i + \epsilon_n, \tag{5.21}$$

where n is a positive integer and $\epsilon_n \to 0$.

5.3. In the theorems about functions of bounded variation we only assume that $\phi(t)$ or $\phi^{(m)}(t)$ is continuous at $t = \pm 1$, but from this we deduce that $\phi(t)$ has continuous variation at those points,† i.e.

$$\lim_{\eta\to 0} \int_{1-\eta}^{1} |d\phi| = 0,$$

or, what is practically the same thing,‡

$$\lim_{\eta\to 0}\lim_{h\to 0} \int_{1-\eta}^{1} \left|\frac{\phi(t)+\phi(t-h)}{h}\right| dt = 0.$$

We now require a similar property for functions which belong to $\text{Lip}(\alpha, 1)$, where $\alpha < 1$. The natural generalization is as follows: $\phi(t) \to 0$ $\text{Lip}(\alpha, 1)$ *as* $t \to a+0$, *if*

$$\phi(t) = o\{(t-a)^{\alpha-1}\}, \tag{5.31}$$

where $0 < \alpha \leqslant 1$, *as* $t \to a+0$, *and*

$$\lim_{\eta\to 0}\lim_{h\to 0} \int_{a}^{a+\eta} \left|\frac{\phi(t+h)-\phi(t)}{h^\alpha}\right| dt = 0. \tag{5.32}$$

Also $\phi(t) \to 0$ $\text{Lip}(q, 1)$, *where* $q = m+\alpha-1$, *if*

$$\phi(a) = \phi'(a) = \ldots = \phi^{(m-2)}(a),$$

and $\phi^{(m-1)}(t) \to 0$ $\text{Lip}(\alpha, 1)$ *as* $t \to a+0$.

It might be supposed that if $\phi(t)$ belongs to $\text{Lip}(\alpha, 1)$, then (5.31) implies (5.32) for $\alpha < 1$ as well as for $\alpha = 1$. But the proof for $\alpha = 1$ depends on the expression of a function of bounded variation as the

† See footnote to § 3.3.

‡ Compare G. H. Hardy and J. E. Littlewood, *Math. Zeit.* 28 (1928), 619.

difference of two monotonic functions. There seems to be no corresponding result for functions which belong to Lip $(\alpha, 1)$, where $\alpha < 1$.

LEMMA C. $\phi(t) \to 0$ Lip $(\alpha, 1)$ *as* $t \to a+0$ *if*

either, (1) $\phi(t)$ *belongs to* Lip* $(\alpha, 1)$ *and satisfies* (5.31),

or, (2) $\phi(t)$ *satisfies* (2) *or* (3) *of Lemma B.*

THEOREM IX. *Suppose that* $\phi(t)$ *is of the form* (1.21) *and satisfies* (5.11). *Suppose further that* $\phi(t)$ *belongs to* Lip $(q+1, 1)$, *and that* $\phi(t) = (1-t^2)^q + \chi(t)$, *where* $\chi(t) \to 0$ Lip $(q+1, 1)$ *as* $t \to \pm 1$. *Then* (i) *and* (ii) *of Theorem III are satisfied, and*

(iii) $N(\rho) = \dfrac{2\rho}{\pi} - (q+1)\log\rho - \log|f(0)| + o(1).$

Again the subtraction of $(1-t^2)^q$ makes no difference if q is 0 or an integer, while for non-integral q, $(1-t^2)^q$ itself does not satisfy (5.31) or (5.32).

5.4. The dominant terms in the asymptotic value for $f(z)$ are given by the following lemma:

LEMMA D. *As* $r \to \infty$

$$s_q(z) = \int_{-1}^{1} e^{zt}(1-t^2)^q\, dt = \frac{\Gamma(q+1)2^q}{z^{q+1}}\{e^z - e^{-z+q\pi i}\} + O\left\{\frac{e^{|x|}}{r^{q+2}}\right\}$$

for $0 \leqslant \theta \leqslant \pi$, *while for* $-\pi \leqslant \theta \leqslant 0$ *we have to read* $e^{-z-q\pi i}$ *in the second term.*

This is derived by easy transformations from well-known formulae for Bessel functions,† since we have

$$s_q(z) = J_{q+\frac{1}{2}}(-iz)\Gamma(q+1)\Gamma(\tfrac{1}{2})(-\tfrac{1}{2}iz)^{-q-\frac{1}{2}}.$$

5.5. *Proof of Theorem VIII.* We have

$$f(z) = s_q(z) + \int_{-1}^{1} e^{zt}\chi(t)\, dt \tag{5.51}$$

$$= s_q(z) + I(z).$$

We define $\chi(t)$ outside the interval $(-1, 1)$ by periodicity. Suppose that $q = m+\alpha-1$; the case $\alpha = 1$ has already been dealt with. For $\alpha < 1$ we have

$$I(z) = \frac{(-1)^m}{z^m} \int_{-1}^{1} e^{zt}\chi^{(m)}(t)\, dt, \tag{5.52}$$

† G. N. Watson, *Theory of Bessel Functions*, Cambridge (1922), 48.

where $\chi^{(m)}(t)$ belongs to Lip* $(\alpha, 1)$. Hence

$$(iy)^m I(iy) = (-1)^m \int_{-1}^{1} e^{iyt}\chi^{(m)}(t)\,dt = (-1)^{m+1}\int_{-1}^{1} e^{iyt}\chi^{(m)}\left(t+\frac{\pi}{y}\right)dt$$

$$= \frac{(-1)^m}{2}\int_{-1}^{1} e^{iyt}\left\{\chi^{(m)}(t) - \chi^{(m)}\left(t+\frac{\pi}{y}\right)\right\}dt.$$

It follows that

$$I(iy) = O\left\{\frac{1}{|y|^m}\int_{-1}^{1}\left|\chi^{(m)}(t) - \chi^{(m)}\left(t+\frac{\pi}{y}\right)\right|dt\right\}$$

$$= o\{|y|^{-m-\alpha}\} = o\{|y|^{-q-1}\}.$$

Also it is easy to see that $I(z) = O(e^{|x|})$ uniformly in θ. Hence applying the theorem of Phragmén and Lindelöf to $z^{q+1}e^{-z}I(z)$ in the right half-plane, and to $z^{q+1}e^{z}I(z)$ in the left half-plane, we have

$$I(z) = o\,(e^{|x|}r^{-q-1})$$

uniformly in θ.

It now follows from Lemma D that

$$z^{q+1}f(z) = \Gamma(q+1)2^q(e^z - e^{-z+q\pi i}) + o\,(e^{|x|})$$

for $y > 0$, and a similar formula holds for $y < 0$. From these we determine the zeros as usual.

5.6. To prove Theorem IX we proceed as above. We deal with the interval $(1-\delta, 1)$ by means of (5.31) and more especially (5.32). The rest of the interval is treated more or less as in § 4.4.

6. We can easily generalize Theorem IV by using a function $\omega(\delta)$ corresponding exactly to the modulus of continuity. Suppose that $\omega(\delta) \neq o\,(\delta^{1-\beta})$, where $0 \leqslant \beta < 1$. Then $\phi(t)$ is said to be continuous (β, ω), if $|\phi(t_2) - \phi(t_1)| \leqslant \delta^\beta\omega(\delta)$, provided that $|t_2 - t_1| \leqslant \delta$. If $\omega(\delta) = o\,(\delta^{1-\beta})$, we use the condition $|\phi'(t_2) - \phi'(t_1)| \leqslant \delta^{\beta-1}\omega(\delta)$ instead. $\phi(t)$ is said to be continuous $(m+\beta, \omega)$ if $\phi^{(m)}(t)$ is continuous (β, ω). Now, if $\phi(t)$ satisfies (1.21) and is continuous (q, ω), we have results like those of Theorem IV. The methods are much the same as those used in Z.

7.1. We shall now consider functions which belong to Lip (α, p),† i.e. functions for which

$$\left\{\int_a^b |\phi(t+h) - \phi(t)|^p\,dt\right\}^{1/p} = O\{h^\alpha\}. \tag{7.11}$$

† G. H. Hardy and J. E. Littlewood, *Math. Zeit.* 28 (1928), 612.

If $\phi(t)$ belongs to $\mathrm{Lip}(\alpha, p)$, where $\alpha p > 1$, then it belongs also to $\mathrm{Lip}(\alpha - 1/p)$.† It follows at once from Theorem XI of Z that when $\phi(t) \to 1$ as $t \to \pm 1$,

$$n(\rho) = 2\rho/\pi + O(\rho^{1-\alpha+1/p}). \tag{7.12}$$

We can, however, obtain a better result.

THEOREM XI. *Suppose that $\phi(t)$ is of the form* (1.21) *with $q = 0$, and that $\phi(t)$ belongs to* $\mathrm{Lip}(\alpha, p)$, *where $\alpha p > 1$, $p > 1$; then*

(i) *all the zeros of $f(z)$ lie in the region $|x| < Kr^{(1-\alpha)p'}$, where*

$$p' = p/(p-1),$$

(ii) $n(\rho) = \dfrac{2\rho}{\pi} + O(\rho^{(1-\alpha)p'})$.

If $\alpha = 1$ and $p > 1$, $\phi(t)$ is an integral;‡ and therefore (5.21) holds. If $p \to \infty$, $p' \to 1$, and the theorem reduces to one for an ordinary Lipschitz condition.§ Since

$$1-\alpha+\frac{1}{p}-(1-\alpha)p' = \frac{\alpha p-1}{p(p-1)},$$

we see that Theorem XI (ii) is a better result than (7.12).

7.2. *Proof of Theorem XI.* We can write

$$f(z) = \int_{-1}^{1} e^{zt}\,dt + \int_{-1}^{1} e^{zt}\chi(t)\,dt = \frac{\sinh z}{z} + I,$$

where $\chi(t)$ belongs to $\mathrm{Lip}(\alpha, p)$. We first find the value of I on the boundary of the region defined in (i), i.e. on

$$x = Kr^{(1-\alpha)p'}. \tag{7.21}$$

The case $\alpha = 1$ has already been proved in Theorem VIII, so that we only have to consider $\alpha < 1$. In this case $x \to \infty$ with r, and also $y/x \to \infty$ with r on (7.21). We shall make use of these points in the proof.

Suppose that $x > 0$, $y > 0$; then on (7.21) we have

$$|I(x+iy)| \leqslant \int_{-1}^{1} e^{xt}\left|\chi(t) - e^{-x\pi/y}\chi\left(t-\frac{\pi}{y}\right)\right|dt + e^{x}\left\{\int_{-1}^{-1+\pi/y} + \int_{1-\pi/y}^{1}\right\}|\chi(t)|\,dt$$

$$= I_1 + I_2.$$

† G. H. Hardy and J. E. Littlewood, loc. cit. 627.

‡ G. H. Hardy and J. E. Littlewood, loc. cit. 27 (1927), 599.

§ Z, Theorems X, XI, and XII.

Since $\chi(t)$ belongs to Lip (α, p), it belongs to Lip $(\alpha-1/p)$; also $\chi(t)\to 0$ as $t\to\pm 1$, so that

$$\chi(t)=O\{(1-t^2)^{\alpha-1/p}\} \tag{7.22}$$

as $t\to 1$. Hence

$$I_2<Ae^x\int\limits_{1-\pi/y}^{1}(1-t)^{\alpha-1/p}\,dt<\frac{Ae^x}{y^{\alpha+1-1/p}}.$$

Again

$$I_1\leqslant\int\limits_{-1}^{1}e^{xt}\left|\chi(t)-\chi\left(t-\frac{\pi}{y}\right)\right|dt+(1-e^{-x\pi/y})\left\{\int\limits_{-1}^{1-\eta}+\int\limits_{1-\eta}^{1}\right\}e^{xt}\left|\chi\left(t-\frac{\pi}{y}\right)\right|dt$$

$$=I_3+I_4+I_5.$$

Using (7.22) we can show† that $I_5<Ae^x/(yx^{\alpha-1/p})$; and it is easy to see that $I_4<Axe^{x(1-\eta)}/y$. Lastly, by Hölder's inequality,

$$I_3<\left\{\int\limits_{-1}^{1}e^{p'xt}\,dt\right\}^{1/p'}\left\{\int\limits_{-1}^{1}\left|\chi(t)-\chi\left(t-\frac{\pi}{y}\right)\right|^p dt\right\}^{1/p}<\frac{Ae^x}{x^{1/p'}y^{\alpha}},$$

where $p'=p/(p-1)$. Combining these results we have

$$|I(x+iy)|<\frac{Ae^x}{x^{1/p'}y^{\alpha}}+\frac{Axe^{x(1-\eta)}}{y}+\frac{Ae^x}{x^{\alpha-1/p}y}<\frac{\frac{1}{2}e^x}{r} \tag{7.23}$$

for every $r>r_0$ on (7.21), provided that K is sufficiently large. Applying the theorem of Phragmén and Lindelöf to $ze^{-z}I(z)$ in the region $x\geqslant Kr^{(1-\alpha)p'}$, we find that (7.23) holds throughout it. Hence

$$zf(z)=e^z(1+\zeta),$$

where $|\zeta|<1$ for $x\geqslant Kr^{(1-\alpha)p'}$. From this (i) follows at once and a similar expression for $x\leqslant -Kr^{(1-\alpha)p'}$; and (ii) follows as usual.

8. When $\phi(t)\sim(1-t)^q(1+t)^{q'}$, where $q>q'$, the typical function is

$$s_{q,q'}(z)=\int\limits_{-1}^{1}e^{zt}(1-t)^q(1+t)^{q'}\,dt,$$

and it is easy to show that

$$s_{q,q'}(z)=\frac{\Gamma(q+1)2^{q'}e^z}{z^{q+1}}(1+\epsilon_r)-(-1)^{q'}\frac{\Gamma(q'+1)2^qe^{-z}}{z^{q'+1}}(1+\epsilon_r).$$

The method used by Hardy‡ then shows that the zeros of $s_{q,q'}(z)$ are given by (1.23).

Near $t=1$ we want hypotheses suitable when $\phi(t)\sim(1-t^2)^q$ and near $t=-1$ those suitable when $\phi(t)\sim(1-t^2)^{q'}$. In the less precise

† Compare (2.12).

‡ G. H. Hardy, *Proc. London Math. Soc.* (2) 2 (1904), 423.

theorems the middle of the interval may be neglected;† otherwise we must assume enough to make the hypotheses join properly.

9.1. Finally we may ask what happens when $\phi(t)\to 0$ very rapidly. There is one function for which calculation is easy.

THEOREM XII. *If* $\phi(t)=(1-t)^{-3/2}e^{-1/(1-t)}$, *then the zeros of* $f(z)$ *are given by*

$$\left.\begin{aligned} x &= (\tfrac{1}{2}n\pi)^{1/2}(1+\epsilon_n) \\ y &= \pm n\pi+(\tfrac{1}{2}n\pi)^{1/2}(1+\epsilon_n). \end{aligned}\right\} \tag{9.11}$$

We have

$$f(z)=\int\limits_{-1}^{1} e^{zt-1/(1-t)}\frac{dt}{(1-t)^{3/2}}=e^z\int\limits_{0}^{2} e^{-zt-1/t}\frac{dt}{t^{3/2}}=e^z\left\{\int\limits_{0}^{\infty}-\int\limits_{2}^{\infty}\right\}e^{-zt-1/t}\frac{dt}{t^{3/2}}.$$

$$=I_1+I_2.$$

It is well known‡ that $I_1=\pi^{1/2}e^{z-2\sqrt{z}}$, and integrating I_2 by parts§ we have

$$f(z)=\pi^{1/2}e^{z-2\sqrt{z}}-\frac{e^{-z-1/2}}{2^{3/2}z}\left\{1+O\left(\frac{1}{r}\right)\right\}$$

uniformly in θ. Hence at a zero

$$e^{2z}=\pi^{-1/2}e^{2\sqrt{z}-1/2}2^{-3/2}z^{-1}\{1+O(r^{-1})\}.$$

Hence by Hardy's method

$$2x=2r^{1/2}\cos\tfrac{1}{2}\theta-\tfrac{1}{2}\log(8\pi e)-\log r+O(r^{-1}),$$
$$2y=\pm 2n\pi+2r^{1/2}\sin\tfrac{1}{2}\theta-\theta+O(r^{-1}).$$

Putting $r=y(1+\epsilon)$, $\theta=\tfrac{1}{2}\pi(1-\epsilon)$, we have (9.11).

9.2. It may be observed that, although a function of the form (1.11) may have a few real zeros, Titchmarsh's|| theorems limit the real parts of the main stream of zeros very strictly. For $\sum|\cos\theta_n|/r_n$ is convergent. This prohibits $x_n=n/\log n$ except for a comparatively small proportion of zeros, although $n(r)\sim 2r/\pi$ permits

$$y_n=\pm n\pi\pm n/\log n \text{ for all } n\geqslant 2.$$

This limitation of the real parts of the zeros is a characteristic of functions which are the sum of exponentials, and by definition $f(z)$ is a limiting case of such a function.

† Z, Theorem XIII.

‡ E. Goursat, *Cours d'Analyse*, 3rd ed., vol. i, p. 295, Ex. 23.

§ Compare G. H. Hardy, *Proc. London Math. Soc.* (2) 2 (1904), 405.

|| E. C. Titchmarsh, *Proc. London Math. Soc.* (2) 25 (1926, Theorems I and IV.

SOME APPLICATIONS OF GENERATING FUNCTIONS TO NORMAL FREQUENCY

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1. Generating functions of frequency and moments

IN many problems of mathematical statistics the moment-generating function is easier to handle than either the frequency function or the frequency-generating function. Indeed, the form of the frequency function can often be deduced by inverting the moment-generating function. The conditions for the validity of this inversion, and for the convergence of moment-generating functions, have been very fully discussed in an important paper* by T. Kameda. No complications arise in the cases treated below, the frequency function being the exponential of a negative definite quadratic form, the character of which may be preserved by a suitable choice of parameters α in forming the generating function.

The intention of the present paper is purely expository, to give some examples of the solution by moment-generating functions of some important problems of normal frequency. No new results are derived, but the systematic use of a determinantal lemma, the extension of which is interesting on its own account, seems to have certain advantages.

The generating functions are defined† thus. If $\phi(x_1, x_2, \ldots, x_n)$, or briefly ϕ, is the frequency function of n variables, then

$$\sum\sum\sum \ldots \phi \, . \, t_1^{x_1} t_2^{x_2} \ldots t_n^{x_n}, \text{ or } \iiint \ldots \phi \, . \, t_1^{x_1} t_2^{x_2} \ldots t_n^{x_n} \, dx_1 dx_2 \ldots dx_n,$$

according as the variables are discrete or continuous, is the generating function of frequency and, with a substitution $t = e^{\alpha}$,

$$\sum\sum\sum \ldots \phi \, . \exp(\textstyle\sum \alpha x), \text{ or } \iiint \ldots \phi \, . \exp(\textstyle\sum \alpha x) \, dx_1 dx_2 \ldots dx_n,$$

is the moment-generating function, since a typical term in α,

$$\alpha_1^{i_1} \alpha_2^{i_2} \ldots \alpha_n^{i_n} / (i_1! \, i_2! \ldots i_n!),$$

has for coefficient

$$m_{i_1, i_2, \ldots, i_n} \equiv \sum\sum\sum \ldots x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \phi,$$

or the analogous integral.

* T. Kameda, *Journ. Fac. Sci. Imp. Univ. Tokyo*, (1) 1 (1925), pp. 1–62, (38).

† H. E. Soper, *Frequency Arrays*, Cambridge, 1922, pp. 8–13.

By the multiplication theorem of frequency, the moment-generating function of the sum of several independent variables, or of independent values of different simultaneous variables, is the continued product of the several moment-generating functions.

To change the origin of measurement of x_r from zero to h_r implies the writing of $t_r^{x_r-h_r}$ for $t_r^{x_r}$ in each term of the frequency-generating function; hence if $F(\alpha)$ is a moment-generating function, the corresponding moment-generating function about the mean $(\bar{x}_1, \bar{x}_2,..., \bar{x}_n)$ is $\exp(-\sum \alpha\bar{x}) . F(\alpha)$.

2. Quadratic forms and a special determinant

From an abstract standpoint the theory of normal distribution in several correlated variables is the theory of the linear transformation of a positive definite quadratic form, such as

$$\sum_{i,j=1}^{n} c_{ij} x_i x_j \equiv C(x,x), \qquad (c_{ij} = c_{ji})$$

in so far as the properties of $\exp\{-\frac{1}{2}C(x,x)\}$, its integral and its moments are affected. Certain standard results are therefore required. First,

$$\iiint_{-\infty}^{\infty} ... \exp\{-\tfrac{1}{2}C(x,x)\}\, dx_1 dx_2 ... dx_n = (2\pi)^{\frac{1}{2}n} C^{-\frac{1}{2}}, \tag{2.1}$$

where $C \equiv |c_{ij}|$ is the determinant of the form $C(x, x)$.

This well-known result follows at once from the fact that when $C(x,x)$ is transformed into a sum of squares $\sum \xi^2$ by a transformation $x_i = \sum_j h_{ij}\, \xi_j$, then $|h_{ij}|^2 = |c_{ij}|^{-1}$; hence the Jacobian $|h_{ij}| = |c_{ij}|^{-\frac{1}{2}}$.

Next, $$C(x+\nu, x+\nu) = C(x,x) + 2\sum \alpha x + C^{-1}(\nu, \nu), \tag{2.2}$$

where $C^{-1}(\alpha, \alpha) = \sum c^{ij}\alpha_i\alpha_j$, and $c^{ij} = C_{ji}/C$. Thus $C^{-1}(\alpha, \alpha)$ is the reciprocal quadratic form, the α's being defined below.

For
$$\begin{aligned} C(x+\nu, x+\nu) &= \sum c_{ij}(x_i+\nu_i)(x_j+\nu_j) \\ &= \sum c_{ij}x_i x_j + 2\sum x_i \sum c_{ij}\nu_j + \sum c_{ij}\nu_i\nu_j \\ &= C(x,x) + 2\sum \alpha x + C^{-1}(\alpha, \alpha), \end{aligned}$$

where $\alpha_i = \sum c_{ij}\nu_j$, for $j = 1, 2,..., n$, and so $\nu_i = \sum c^{ij}\alpha_j$.

The point of this is that just as in elementary algebra we 'complete the square' of $cx^2 + 2\alpha x$ by adding $c^{-1}\alpha^2$, so a quadratic form plus linear terms, $C(x,x) + 2\sum \alpha x$, may be 'completed' by adding $C^{-1}(\alpha, \alpha)$.

Lastly, the values of two related types of determinant will be required,

$$\begin{vmatrix} a+b & b & b & \dots \\ b & a+b & b & \dots \\ b & b & a+b & \dots \\ . & . & . & \dots \end{vmatrix}_n \quad (2.3) \text{ and } \quad \begin{vmatrix} A+B & B & B & \dots \\ B & A+B & B & \dots \\ B & B & A+B & \dots \\ . & . & . & \dots \end{vmatrix} \quad (2.4).$$

In (2.3) the diagonal element is always $a+b$, and every other element is b. The operations 'row 1 — row 2, row 2 — row 3,...,' and then 'col n + col 1 + col 2 +...+ col $\overline{n-1}$' reduce it to a form in which its value* is seen to be $a^{n-1}(a+nb)$. In (2.4) the A and B represent square m-by-m blocks of elements, the block $A+B$ being obtained by adding corresponding elements of A and B. The procedure for evaluation is exactly similar to that for (2.3), rows and columns of blocks being subtracted or added like elements. Laplacian expansion then gives the value of (2.4), if of order mn, as $|A|^{n-1}.|(A+nB)|$.

3. Normal frequency in many variables

We proceed to investigate the simultaneous frequency-function of variables $x_1, x_2,..., x_n$, where each x_i, measured as a deviation from its mean value, arises as the sum of N elements δx_i, each simultaneous set of elements being subject to a law of frequency. The law is taken to be of such a kind that the mean absolute value of $|\delta x_i|$ is of order N^{-1}, the mean values of $(\delta x_i)^2$ and $\delta x_i \delta x_j$ of order N^{-2}, while the cubic and later elementary moments are of higher order. These conditions are satisfied by a very wide class of laws of frequency, but not by all; for example, in the Poisson case, where the probability of non-zero deviations is itself of order N^{-1}, the elementary moments are of higher infinitesimal order than the above.

The moment-generating function of $x_1, x_2,..., x_n$ will be the continued product of N elementary moment-generating functions, and so the second or quadratic moments $r_{ij}\sigma_i\sigma_j$ are of order N^{-1}, the cubic and later moments of higher order. The moment-generating function

$$1+\sum r_{ij}\sigma_i\sigma_j\alpha_i\alpha_j/2!+... \qquad (r_{ii}=1,\ r_{ij}=r_{ji}).$$

is therefore replaced, to the same order in N, by $\exp\{\tfrac{1}{2}R(\sigma\alpha, \sigma\alpha)\}$, where $R=|r_{ij}|$; and the continuous frequency-function ϕ which has this moment-generating function is given by

$$\exp\{\tfrac{1}{2}R(\sigma\alpha,\sigma\alpha)\} = \iiint_{-\infty}^{\infty} ... \phi . \exp(\sum \alpha x)\, dx_1 dx_2 ... dx_n.$$

* The determinant is well known and is usually evaluated indirectly by considerations which, however, do not apply at once to (2.4).

The solution of this equation, by (2.1) and (2.2), is

$$\phi = (2\pi)^{-\frac{1}{2}n}|R|^{-\frac{1}{2}}\sigma_1\sigma_2\ldots\sigma_n\exp\{-\tfrac{1}{2}R^{-1}(x/\sigma, x/\sigma)\},$$

and this is the Edgeworth-Pearson function of normal correlation. It arises also if the $x_1, x_2,\ldots, x_n$ are linearly correlated* through being each of them linear functions of a larger number of uncorrelated variables $\xi_1, \xi_2,\ldots, \xi_s$, normally distributed.

4. Distribution of normal variance

For the next example we may investigate the distribution of s^2, the variance or squared standard deviation of x, as calculated from samples of N values drawn from an infinite normal stock with frequency-function $(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}$. Following Wishart† we shall denote s^2 by a_{11}.

A statistical parameter θ will in general itself possess‡ a moment-generating function

$$F(\theta,\alpha) = \int \phi(\theta)e^{\alpha\theta}\,d\theta,$$

where, if θ is calculated from independent sample values $x_1, x_2,\ldots, x_n$, the function $\phi(\theta)$ will represent the compound frequency of the sample values, and the integral will be N-ple. Thus in the present case the variance about the mean of sample is

$$a_{11} = \sum x^2/N-(\sum x)^2/N^2,$$

and so $F(a_{11},\alpha)$

$$= (2\pi)^{-\frac{1}{2}N}\int\!\!\int\!\!\int_{-\infty}^{\infty}\ldots\exp(-\tfrac{1}{2}\sum x^2)\exp\{\alpha\sum x^2/N-\alpha(\sum x)^2/N^2\}dx_1dx_2\ldots dx_N.$$

The determinant of the quadratic form under the exponential is of the type (2.3), with $1-2\alpha/N$ for a, $2\alpha/N^2$ for b. Its value is thus found at once to be $(1-2\alpha/N)^{N-1}$, and so, by (2.1), we have

$$F(a_{11},\alpha) = (1-2\alpha/N)^{-\frac{1}{2}(N-1)} = 1+(N-1)\alpha/N+\ldots.$$

The coefficient of α puts in evidence the familiar result that the mean of s^2 for samples of N is $(N-1)\sigma^2/N$, where σ^2 is the variance of the sampled population. Also the moment-generating function of

* M. J. van Uven, *Proc. Amsterdam Acad.* 16 (1214) pp. 8–13.

† J. Wishart, *Biometrika*, XXA (1928) pp. 32–52. Also V. Romanovsky, *Comptes Rendus*, 180 (1925), 1897; *Metron*, 5 (1925), 27, 32, 35–6.

‡ Kameda, loc. cit.

s^2 about its own mean is then

$$(1-2\alpha/N)^{-\frac{1}{2}(N-1)}.\exp\{-(N-1)\alpha/N\},$$

from which its own variance and higher moments are easily calculated.

5. Product-moment in bivariate normal sample

We next consider N sample drawings from an infinite normal bivariate population with frequency-function

$$(2\pi)^{-1}(1-\rho^2)^{-\frac{1}{2}}\exp\{-\frac{\frac{1}{2}}{1-\rho^2}(x^2-2\rho xy+y^2)\},$$

and suppose the product-moment rs_1s_2, or a_{12}, to be calculated about the means as

$$a_{12}=\sum xy/N-(\sum x)(\sum y)/N^2.$$

Thus the moment-generating function, say $F(a_{12}, \alpha)$, is the $2N$-ple integral of

$$(2\pi)^{-N}(1-\rho^2)^{-\frac{1}{2}N}\exp\left\{-\frac{\frac{1}{2}}{1-\rho^2}\sum(x^2-2\rho xy+y^2)\right\}\times$$

$$\times\exp\{\alpha\sum xy/N-\alpha(\sum x)(\sum y)/N^2\}.$$

The determinant of the quadratic form in the integrand now belongs to the type (2.4), where the arrays A, B are seen to be

$$A=\begin{bmatrix} \dfrac{1}{1-\rho^2} & -\dfrac{\rho}{1-\rho^2}-\dfrac{\alpha}{N} \\ -\dfrac{\rho}{1-\rho^2}-\dfrac{\alpha}{N} & \dfrac{1}{1-\rho^2} \end{bmatrix}, \quad B=\begin{bmatrix} 0 & \alpha/N^2 \\ \alpha/N^2 & 0 \end{bmatrix}.$$

Hence

$$F(a_{12},\alpha)=(1-\rho^2)^{\frac{1}{2}(N-1)}\begin{vmatrix} 1 & -\rho-\dfrac{\alpha}{N}(1-\rho^2) \\ -\rho-\dfrac{\alpha}{N}(1-\rho^2) & 1 \end{vmatrix}^{-\frac{1}{2}(N-1)}$$

$$=\{1-2\rho\alpha/N-\alpha^2(1-\rho^2)/N^2\}^{-\frac{1}{2}(N-1)}.$$

The mean value of rs_1s_2 as calculated from samples of N is seen to be $(N-1)\rho\sigma_1\sigma_2/N$, and the moment-generating function about this mean is

$$\exp\{-(N-1)\rho\alpha/N\}.\{1-2\rho\alpha/N-\alpha^2(1-\rho^2)/N^2\}^{-\frac{1}{2}(N-1)}.$$

6. Simultaneous quadratic moments in sample

The two preceding examples indicate a general process and result.

Samples of N are taken from a multivariate normal population with frequency-function

$$(2\pi)^{-\frac{1}{2}n}|\rho_{ij}|^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}R^{-1}(x,x)\},$$

where $R(x,x)=\sum_{i,j=1}^{n}\rho_{ij}x_i x_j$, and $R^{-1}(x,x)$ is the reciprocal form. Quadratic moments $r_{ij}s_i s_j$, or a_{ij}, are calculated by means of

$$a_{ij}=\sum' x_i x_j/N-(\sum' x_i)(\sum' x_j)/N^2,$$

where $\sum'$ is used to denote summation over the N sample values of particular variables, and $\sum$ is reserved for summation over the n different variables.

The simultaneous moment-generating function of sample quadratic moments, $F(a,\alpha)$, is thus the nN-ple integral of

$$(2\pi)^{-\frac{1}{2}nN}|\rho_{ij}|^{-\frac{1}{2}N}\exp\{-\tfrac{1}{2}\sum' R^{-1}(x,x)\}\times$$
$$\times\exp\{N^{-1}\sum'\sum\alpha_{ij}x_i x_j-N^{-2}\sum\alpha_{ij}(\sum' x_i)(\sum' x_j)\}.$$

Once again the determinant of the quadratic form involved, of order nN, is of type (2.4), the arrays A, B being

$$A=[\rho^{ij}]-\frac{1}{N}\begin{bmatrix}2\alpha_{11} & \alpha_{12} & \cdots \\ \alpha_{12} & 2\alpha_{22} & \cdots \\ \cdot & \cdot & \cdots\end{bmatrix},\qquad B=\frac{1}{N^2}\begin{bmatrix}2\alpha_{11} & \alpha_{12} & \cdots \\ \alpha_{12} & 2\alpha_{22} & \cdots \\ \cdot & \cdot & \cdots\end{bmatrix}.$$

If the array of α_{ij}'s be denoted by $[\alpha_{ij}]$, with the convention that diagonal elements are doubled, then by (2.4)

$$\begin{aligned}F(a,\alpha) &= |\rho_{ij}|^{-\frac{1}{2}N}|[\rho^{ij}-N^{-1}\alpha_{ij}]|^{-\frac{1}{2}(N-1)}|\rho^{ij}|^{-\frac{1}{2}} \\ &= |(1-N^{-1}[\rho_{ij}][\alpha_{ij}])|^{-\frac{1}{2}(N-1)}.\end{aligned}$$

The above denotes a power of a determinant in which the elements are those of the product $-[\rho_{ij}][\alpha_{ij}]$, multiplied by N^{-1} and increased by unity in the diagonal. The result, allowance being made for difference in notation, is that first given* in 1928 by Dr. J. Wishart, who obtained it as a deduction from his exact form for the simultaneous frequency distribution of these quadratic moments.

The corresponding distributions, not about the mean of sample but about the mean of the population, are obtained at once by suppressing the terms in $1/N^2$ in the formulae for calculating the moments. The effect is to replace the elements b or arrays B in the determinants by zeros, so that the power of the resulting determinant appears finally as $-\frac{1}{2}N$ instead of $-\frac{1}{2}(N-1)$.

* Wishart, loc. cit.

SIMULTANEOUS QUADRATIC AND LINEAR REPRESENTATION

By GORDON PALL

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We investigate the solvability in integers $x_1, ..., x_s$ of the simultaneous equations

$$a = \sum a_{ij} x_i x_j \equiv f(x_1, ..., x_s), \qquad (1)$$
$$b = \sum b_i x_i \quad \equiv l(x_1, ..., x_s),$$

where f and l are respectively quadratic and linear forms with integral coefficients, and

$$a_{ij} = a_{ji} \quad (i,j = 1,..., s), \qquad D \equiv |a_{ij}| \neq 0. \qquad (2)$$

The special equations

$$a = x_1^2 + ... + x_s^2, \qquad b = x_1 + ... + x_s \qquad (3)$$

are studied in detail (sections 2, 4, and 5).

It is seen that there exists a (1, 1) correspondence between the integral solutions $(x_1,..., x_s)$ of (1) and the integral solutions $(y_2,..., y_s)$ of an equation

$$k(Ka - Db^2) = \psi(y_2,..., y_s) \qquad (4)$$

and a system S of linear congruences in the y_h and b. Here ψ is a quadratic form with integral coefficients, k is a constant, and

$$K \equiv F(b_1,..., b_s), \qquad (5)$$

where F is the adjoint of f, is supposed not zero.

The system S can be reduced to simplest terms, and in special cases can be entirely eliminated by using the signs of the y_h or other transformations of ψ.

In section 3 we shall find a method of obtaining a set of sufficient conditions for the solvability of (1) in integers $x_i \geqslant 0$, when f is positive definite.

An interesting way of expressing the above-mentioned (1, 1) correspondence is illustrated in Theorem 2.

In section 5 we shall, for any $s \geqslant 8$, determine all pairs (a, b) of integers such that equations (3) are solvable in integers x_i.

1. Let F be the adjoint form of f. Then

$$\det(\tau a_{ij} - b_i b_j) = \tau^s D - \tau^{s-1} F(b_1,..., b_s). \qquad (6)$$

Hence the determinant of the form

$$\phi(x_1,..., x_s) \equiv Kf - Dl^2 \qquad (7)$$

is zero. By rational operations we can express it in the form

$$k\phi = \sum c_h X_h^2 \equiv \psi(X_2, \dots, X_s), \tag{8}$$

$$X_h = c_{h1}x_1 + c_{h2}x_2 + \dots + c_{hs}x_s \quad (h = 2, \dots, s),$$

where k and all the c's are integers, $k > 0$.

By (7), (8), and (1_2),

$$kKf = kD(\sum b_i x_i)^2 + c_2(\sum c_{2i}x_i)^2 + \dots + c_s(\sum c_{si}x_i)^2. \tag{9}$$

The Hessian being an invariant of weight two,

$$(kK)^s D = kDc_2c_3 \dots c_s \Delta^2, \tag{10}$$

where $c_{1j} \equiv b_j$ $(j = 1, \dots, s)$, $\Delta \equiv |c_{ij}|$. If $K \neq 0$, Δ and $c_2, \dots, c_s$ are not zero. Hence the equations

$$\begin{gathered} b_1x_1 + \dots + b_sx_s = b, \\ c_{h1}x_1 + \dots + c_{hs}x_s = y_h \quad (h = 2, \dots, s) \end{gathered} \tag{11}$$

have the unique solutions

$$x_j = (C_{1j}b + C_{2j}y_2 + \dots + C_{sj}y_s)/\Delta, \tag{12}$$

where C_{ij} is, naturally, the cofactor of c_{ij}.

THEOREM 1. *Let f and l be as in the introductory paragraph, and let $F(b_1, \dots, b_s) \neq 0$. We can set up identities* (8) *by simple operations. For any such identity there is a* $(1, 1)$ *correspondence between the sets of integers x_i satisfying* (1) *and the sets of integers y_h satisfying* (4) *together with a system S of linear congruences involving the y_h and b.*

Evidently S is any system equivalent, under the hypothesis that the y_h satisfy (4), to

$$C_{1j}b + C_{2j}y_2 + \dots + C_{sj}y_s \equiv 0 \pmod{\Delta} \quad (j = 1, \dots, s). \tag{13}$$

2. For (3), one expression (8) is

$$(s-1)!(sa - b^2) = \sum_{h=2}^{s} \frac{s!}{h(h-1)} X_h^2, \tag{14}$$

where $\quad X_h = (h-1)x_{s-h+1} - x_{s-h+2} - \dots - x_s.$

As generally for (1), the identity (14) for (3), obtained by completing squares and preserving the integrality of coefficients by adequate multipliers, can be greatly simplified by using relations between the X_h. For example, the identity for $s = 4$, namely,

$$3(4a - b^2) = X_4^2 + 2X_3^2 + 6X_2^2,$$

where $X_4 = X_3 + 3Y$, $Y \equiv x_1 - x_2$, readily reduces to

$$4a - b^2 = (x_1 + x_2 - x_3 - x_4)^2 + 2(x_3 - x_4)^2 + 2(x_1 - x_2)^2, \tag{15}$$

and hence, by way of the almost trivial identity

$$2\xi^2 + 2\eta^2 = (\xi + \eta)^2 + (\xi - \eta)^2, \tag{16}$$

to the classical identity

$$4a-b^2 = (x_1+x_2-x_3-x_4)^2+(x_1-x_2+x_3-x_4)^2+(x_1-x_2-x_3+x_4)^2. \tag{17}$$

In a similar way, for $s = 3, 5, 6$ we get

$$\tfrac{1}{2}(3a-b^2) = (x_1-x_2)^2+(x_1-x_2)(x_2-x_3)+(x_2-x_3)^2; \tag{18}$$

$$4(5a-b^2) = (4x_1-x_2-x_3-x_4-x_5)^2+5(x_2+x_3-x_4-x_5)^2+ \\ +5(x_2-x_3+x_4-x_5)^2+5(x_2-x_3-x_4+x_5)^2; \tag{19}$$

$$2(6a-b^2) = (2x_1+2x_2-x_3-x_4-x_5-x_6)^2+3(x_3+x_4-x_5-x_6)^2+ \\ +6(x_1-x_2)^2+6(x_3-x_4)^2+6(x_5-x_6)^2. \tag{20}$$

A repeated use of (16) gives a simple identity for

$$s(x_1^2+...+x_s^2) \qquad (s = 2^h)$$

as a sum of 2^h squares $(x_1+e_2x_2+...+e_sx_s)^2$, where the e_h are ± 1.

Let $N_s(a,b)$ denote the number of representations of the integer a as a sum of s squares the sum of whose bases is b; that is, the number of sets $(x_1,..., x_s)$ satisfying (3).

THEOREM 2. *Let a and b be integers. Then*

(A) $N_3(a,b) = \omega_1 N(6a-2b^2 = x^2+3y^2)$
$= \omega_2 N(3a-b^2 = 2x^2+2xy+2y^2)$;

(B) $N_4(a,b) = \omega_3 N(4a-b^2 = x^2+y^2+z^2)$
$= \omega_4 N(4a-b^2 = x^2+2y^2+2z^2)$;

(C) $N_5(a,b) = \omega_5\{N(5a-b^2 = x^2+5y^2+5z^2+5w^2)+$
$+N(20a-4b^2 = x^2+5y^2+5z^2+5w^2)\}$

where $\omega_1 = \omega_2 = 1 \quad (3|b)$, $\qquad \omega_3 = \omega_4 = 1 \quad (a \equiv b \equiv 0)$,
$= \tfrac{1}{2} \quad (3\nmid b)$; $\qquad = \tfrac{1}{2} \quad (a \equiv b \equiv 1)$,
$\omega_5 = \tfrac{1}{2}\ (5|b, a \equiv b)$, $\qquad = 0 \quad (a \not\equiv b)$;
$= \tfrac{1}{4}\ (5\nmid b, a \equiv b)$,
$= 0\ (a \not\equiv b)$, mod 2.

A. Let $s = 3$. Equations (3) are algebraically equivalent to (18) and (3_2). Hence there is a $(1,1)$ correspondence between the sets of integers x_i satisfying (3) and the sets of integers y_h satisfying

$$\tfrac{1}{2}(3a-b^2) = y_2^2+y_2y_3+y_3^2, \qquad b \equiv y_2-y_3 \pmod 3. \tag{21}$$

Now by (21_1), $b^2 \equiv (y_2-y_3)^2 \pmod 3$. Hence if $3|b$, (21_2) holds for every set of y_h of (21_1); if $3\nmid b$ it holds for precisely half of these sets since but one of y_2-y_3, y_3-y_2 is congruent to $b \pmod 3$. Hence we have the value of ω_2. But $\omega_1 = \omega_2$, since

$$N(4n = x^2+3y^2) = N(n = x^2+xy+y^2).$$

B. Let $s = 4$. On using (17) in place of (18), we see that there is a $(1, 1)$ correspondence between the x_i of (3) and the y_h of

$$4a-b^2 = y_2^2+y_3^2+y_4^2, \qquad y_2+y_3+y_4+b \equiv 0 \pmod 4. \tag{22}$$

For later use we note that the correspondence is given by

$$x_1 = \tfrac{1}{4}(b+y_2+y_3+y_4), \qquad x_2 = \tfrac{1}{4}(b+y_2-y_3-y_4), \text{ etc.} \tag{23}$$

If $a \not\equiv b \pmod 2$, (3) are of course not solvable. If a, b are odd, (22_1) implies y_2, y_3, y_4 odd, whence (22_2) holds for precisely half the solutions of (22_1), just one of $b+y_2+y_3\pm y_4$ being divisible by 4. Now $c^2 \equiv 2c \pmod 8$ if c is even. Hence, if a, b are even, (22_1) implies

$$-2b \equiv 2y_2+2y_3+2y_4 \pmod 8$$

and hence (22_2). Thus we have ω_3. But $\omega_4 = \omega_3$, since

$$N(n = x^2+2y^2+2z^2) = N(n = x^2+y^2+z^2) \text{ if } n \equiv 0, 3, 4, \pmod 8.$$

C. Let $s = 5$. Using (19), we find a $(1, 1)$ correspondence between the solutions x_i of (3) and y_h of

$$4(5a-b^2) = y_2^2+5y_3^2+5y_4^2+5y_5^2, \tag{24}$$

$$y_2 \equiv b \pmod 5, \; y_2+y_3+y_4+y_5 \equiv 0 \pmod 4.$$

The rest of the argument is left to the rèader.

3. Inequalities; solution in integers $x_i \geqslant 0$

Let f and l be as in section 1 and let f be positive definite. Then $K > 0$, $D > 0$, $c_2,..., c_s > 0$, and so

$$Kf \geqslant Dl^2 \text{ for all real } x_i. \tag{25}$$

As a corollary, if f is positive the inequality $Ka \geqslant Db^2$ is necessary for the solvability of (1) in real x_i. Suppose $b_i \geqslant 0$ $(i = 1,..., s)$, and

$$f = a_1 x_1^2+...+a_s x_s^2. \tag{26}$$

The assumption $b \geqslant 0$, $x_i \leqslant -1$, for some i, implies

$$(b+b_i)^2 \leqslant \tau_i(a-a_i), \qquad \tau_i \equiv F(b_1,..., b_{i-1}, 0, b_{i+1},..., b_s)/D,$$

by use of (25). Hence, if f and l are of the above form, the x_i of every solution of (1) in integers are all non-negative if

$$b \geqslant 0, \qquad (b+b_i)^2 > \tau_i(a-a_i) \; (i = 1,..., s). \tag{27}$$

To get a set of sufficient conditions for solvability of (1) in integers $x_i \geqslant 0$, merely adjoin (27) to a set of necessary and sufficient conditions for solvability in integers x_i. That (27) may sometimes be improved on is illustrated in the next section.

4. Conditions for solvability of (3) in non-negative integers

By Theorem 2, (A), equations

$$a = x^2+y^2+z^2, \qquad b = x+y+z \tag{28}$$

are solvable in integers x, y, z, if and only if

$$\tfrac{1}{2}(3a-b^2) \text{ is represented in } \xi^2+3\eta^2. \tag{29}$$

By section 3, each of $x, y, z \geqslant 0$, if $b \geqslant 0$, $b^2+2b+3 > 2a$.

THEOREM 3. *If $s = 4$ the conditions*

$$a \equiv b \pmod 2, \qquad 4a-b^2 = \textit{a sum of three squares}, \tag{30}$$

are necessary and sufficient for the solvability of (3) *in integers x_i. No x_i of any solution is negative if*

$$b \geqslant 0, \qquad b^2+2b+4 > 3a. \tag{31}$$

A solution in integers $x_i \geqslant 0$ exists if (30_2), (31_1), *and*

$$a, b \textit{ are even}, \qquad 3b^2+8b+16 > 8a. \tag{32}$$

Only the last statement is new.* Its proof is clear from (22) and (23). When a, b are even, the signs of the y_h are all at our disposal, and, given a representation (22_1), we can suppose $y_2 \leqslant y_3 \leqslant -y_4$, $y_4 \geqslant 0$. Then the x_i of (23) are all greater than -1 if

$$-y_2-y_3-y_4 < b+4,$$

and hence, since

$$(-y_2-y_3-y_4)^2 \leqslant (|y_2|+|y_3|)^2 \leqslant 2(y_2^2+y_3^2+y_4^2) = 2(4a-b^2),$$

if $2(4a-b^2) < (b+4)^2$, which is (32_2).

THEOREM 4. *If $s = 5$, 6, or 7, equations* (3) *are solvable in integers x_i, if and only if*

$$a \equiv b \pmod 2, \qquad sa \geqslant b^2. \tag{33}$$

They are solvable in integers $x_i \geqslant 0$, if also

$$b \geqslant 0, \qquad b^2 \geqslant 3a-5. \tag{34}$$

I. *Proof for $s = 5$.* We use the fact that an integer is a sum of three squares, if and only if it is positive or zero and not of the form $4^h(8v+7)$, to show that $5a-b^2$ is represented in $x^2+5(y^2+z^2+w^2)$, if and only if $5a \geqslant b^2$. Hence the first part of Theorem 4 for $s = 5$ follows from Theorem 2, (C).

* Cf. Cauchy, *Œuvres*, (2), vol. 6, pp. 320–53; Legendre, *Théorie des Nombres*, ed. 3, vol. ii, Nos. 624–30. Many elaborations and extensions of Theorem 3, beyond those of the present section, are to be found in the writer's paper 'On sums of two or four values of a quadratic function', to appear shortly in some American journal.

If $(b+1)^2 > 4(a-1)$, $b \geqslant 0$, then all x_i are positive or zero by the preceding section. Hence assume

$$3a-5 \leqslant b^2 \leqslant 4(a-1), \qquad b \geqslant 0. \tag{35}$$

If a, b are odd, take $x_5 = 0$; if a, b are even, take $x_5 = 1$. By (35) all conditions of Theorem 3 are satisfied with $a-x_5^2$, $b-x_5$ in place of a, b.

II. Given that equations (3) are solvable in integers when (33) holds, [non-negative integers, if (34) holds too] for $s = S-1 \geqslant 5$, we wish to examine its like solvability for $s = S$, if

$$(S-1)a < b^2 \leqslant Sa, \qquad (b \geqslant 0). \tag{36}$$

We have to seek a non-negative integer x such that

$$\phi(x) \geqslant 0, \qquad \phi(x) \equiv (S-1)(a-x^2)-(b-x)^2. \tag{37}$$

This must hold for x the integer nearest b/S, which is the turning-point of the parabola $z = \phi(y)$. The only case of doubt is therefore $\phi(\pm\frac{1}{2}+b/S) < 0$, whence a and b satisfy

$$Sa \geqslant b^2 > Sa-\tfrac{1}{4}S^2/(S-1). \tag{38}$$

If $S = 6$ or 7, (38) becomes $Sa = b^2$, since $Sa-1$ is not a square. Then $b = Sx$, $a = Sx^2$, and (3) are solvable with $x_i = x$. This completes the proof of Theorem 4.

If $S = 8$, (38) reduces to $b^2 = 8a$. Then $b \equiv 0 \pmod 4$. If $b = 8x$, then $a = 8x^2$ and (3) are solvable as before. But if $b = 4y$ (y odd), then $a = 2y^2$, and equations (3) for $s = 8$ are not solvable in integers x_i. For (37_1) holds for no integer x in view of $\phi(\frac{1}{2}y+\frac{1}{2}) = -2$.

5. Case $s \geqslant 8$; all (a, b) satisfying (3)

Let (3_S) denote equations (3) for $s = S$.

LEMMA 1. *Let $s \geqslant 8$. Let a_0, b_0 be integers of the same parity for which equations (3_s) are* (i) *solvable,* (ii) *not solvable, in integers x_i. Write* $r = sa_0-b_0^2$, $\rho =$ *least absolute residue of b_0* $(\operatorname{mod} s)$. (39)
Then equations (3_s) are (i) *solvable,* (ii) *not solvable, in integers x_i for every pair (a, b) of integers satisfying* (33_1) *and*

$$sa-b^2 = r, \qquad b \equiv \rho \pmod s. \tag{40}$$

For any integers r, ρ, s, $(s \geqslant 8)$, let $(r, \rho)_s$ denote the set of all pairs (a, b) of integers of the same parity satisfying (40). This will usually be a null class (compare Lemma 2 following). The set of all $(a+x^2, b+x)$, for all (a, b) of $(r, \rho)_s$ and all integers x, coincides with the set of all (a', b') of like parity satisfying

$$(s+1)a'-b'^2 = r', \qquad b' \equiv \rho' \pmod{s+1} \tag{41}$$

for *any* integer v, where

$$r' = r+\{r+(sv-\rho)^2\}/s, \qquad \rho' = \rho+v. \tag{42}$$

To see this we calculate $(s+1)(a+x^2)-(b+x)^2$ with $x = v+(b-\rho)/s$. The sets $(r',\rho')_{s+1}$ are said to *derive* from $(r,\rho)_s$.

Now all the pairs (a,b) of integers of like parity can be segregated into sets of the form $(r,\rho)_s$. Call a set $(r,\rho)_s$ *solvable* if (3_s) are solvable in integers x_i for all the elements (a,b) of $(r,\rho)_s$; *unsolvable* if (3_s) are unsolvable for all (a,b) of $(r,\rho)_s$. For $s = 8$ every set $(r,\rho)_8$ is either solvable or unsolvable; indeed $(0,0)_8$ and all $(r,\rho)_8$ with $r > 0$ are solvable, and $(0,\pm 4)_8$ and all $(r,\rho)_8$ with $r < 0$ are unsolvable.

Every non-null set $(r',\rho')_{s+1}$ derives from an infinity of sets $(r,\rho)_s$, all obtained by solving (42) for r and ρ. Every derived set of a solvable set is solvable. From the nature of the transformation

$$(a' = a+x^2,\ b' = b+x;\ \textit{all } x)$$

it is now clear that, if every set $(r,\rho)_s$ is either solvable or unsolvable, the same is true of every set $(r',\rho')_{s+1}$. From the fact for $s = 8$, every set $(r,\rho)_s$, $s \geqslant 8$, is solvable or unsolvable. This is the lemma.

LEMMA 2. *A set $(r,\rho)_s$ is non-null, if and only if*

$$r+\rho^2-\rho s \equiv 0 \pmod{2s}. \tag{43}$$

For this is a necessary and sufficient condition for the existence of an integer t such that

$$\frac{(\rho+ts)^2+r}{s} \text{ is an integer of the same parity as } \rho+ts.$$

LEMMA 3. *Let r, ρ, s be integers satisfying* (43), $s \geqslant 8$, *and* $|\rho| \leqslant \frac{1}{2}s$. *Then the following three propositions are equivalent*:

$$\textit{the set } (r,\rho)_s \textit{ contains an element } (a,b) \textit{ with } a < b; \tag{44}$$

$$\textit{the set } (r,\rho)_s \textit{ is unsolvable}; \tag{45}$$

$$r < |\rho| s-\rho^2. \tag{46}$$

It is trivial that (44) implies (45). For (3_s) imply

$$a \geqslant \sum |x_i| \geqslant b.$$

It is trivial that (46) implies (44). For let $b = \rho$ if $\rho \geqslant 0$, $b = \rho+s$ if $\rho < 0$. Then $a = c$ or $c+2\rho+s$, and $a < b$, where

$$c = (r+\rho^2)/s. \tag{47}$$

We prove by induction on s that (45) implies (46). We know this

to be true if $s=8$. Assume for a given $s \geqslant 8$ that the unsolvable non-null sets are precisely those satisfying

$$(43),\ (46),\ \text{and}\ |\rho| \leqslant \tfrac{1}{2}s. \tag{48}$$

The sets $(r,\rho)_s$ from which $(r',\rho')_{s+1}$ derives are, as we see by inverting (42), given by

$$r = r' - \frac{r' + \{(s+1)v - \rho'\}^2}{s+1} \tag{49}$$

where $v = 0, \pm 1, \pm 2, \ldots$. The set $(r',\rho')_{s+1}$ is unsolvable, if and only if (46) holds with r in (49) and with τ_v in place of ρ, for every v. Here τ_v denotes the least absolute residue of $\rho' - v$ modulo s. Hence $(r',\rho')_{s+1}$ unsolvable implies

$$r' - \{(s+1)v - \rho'\}^2/s < (s+1)|\tau_v| - (s+1)\tau_v^2/s \tag{50_v}$$

for every v. If $|\rho'| \leqslant \frac{1}{2}s$, (50_0) is

$$r' < (s+1)|\rho'| - \rho'^2. \tag{51}$$

If $\rho' = \frac{1}{2}(s+1)$, (50_1) is equivalent to (51). The induction is complete.

As a consequence of these lemmas we have

THEOREM 5. *Let $s \geqslant 8$. For each of $\rho = 3, 4, \ldots, [\frac{1}{2}s]$ determine all integers $r \geqslant 0$ such that*

$$\frac{r+\rho^2}{s} < \rho, \qquad \frac{r+\rho^2}{s} \equiv \rho \pmod 2. \tag{52}$$

For any such ρ, r and any a, b satisfying

$$b \equiv \pm\rho \pmod s, \qquad sa - b^2 = r, \tag{53}$$

the equations (3) *are not solvable in integers x_i. They are so solvable for every other pair a, b satisfying* (33).

The largest r is $[\frac{1}{4}(s^2 - 8s)]$. All values (ρ, r) for some small values s follow:

$s = 8$: (4,0);

$s = 9$: (3,0), (4,2);

$s = 10$: (3,1), (4,4), (5,5);

$s = 11$: (3,2), (4,6), (5,8);

$s = 12$: (3,3), (4,8), (5,11), (6,12);

$s = 16$: (3,7), (4,16), (5,23), (6,28), (7,31), (8,32), (8,0).

INTEGRALS EXPRESSING PRODUCTS OF BESSEL'S FUNCTIONS

By T. W. CHAUNDY

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1. Mellin's principle

THERE is a principle due to Mellin* which enables us to obtain many of the standard integrals in the theory of Bessel's functions (and elsewhere) in a systematic and comparatively simple fashion. Since I cannot find reference to it in Watson's *Bessel's Functions*, it may be that this method deserves to be better known.

The integrals contemplated by Mellin are essentially of the type

$$y = \int \phi\left(\frac{z}{t}\right)\psi(t)\,\frac{dt}{t}, \tag{1}$$

taken either between limits independent of z or round a closed contour.

I write δ for the operator zd/dz. Then

$$\begin{aligned}\delta y &= \int \phi'\left(\frac{z}{t}\right)\psi(t)\frac{z}{t^2}\,dt \\ &= -\left[\phi\left(\frac{z}{t}\right)\psi(t)\right] + \int \phi\left(\frac{z}{t}\right)\{t\psi'(t)\}\,\frac{dt}{t}\end{aligned} \tag{2}$$

on integration by parts. It is convenient to write ζ for z/t, to denote the operators in ζ, t corresponding to δ by δ_ζ, δ_t, and to regard these operators as essentially *partial* differential operators operating only on the functions of ζ, t respectively to which they are attached. We may therefore write (2) as

$$\delta \int \phi(\zeta)\psi(t)\,\frac{dt}{t} = \int \phi(\zeta)\delta_t\psi(t)\,\frac{dt}{t} - [\phi(\zeta)\psi(t)],$$

where the brackets [ ] indicate terms evaluated at the limits. Since $\delta\phi(\zeta)$ is $\delta_\zeta\phi(\zeta)$, we have by repeated applications of the above process

$$\delta^n \int \phi(\zeta)\psi(t)\,\frac{dt}{t} = \int \phi(\zeta)\delta_t^n\psi(t)\,\frac{dt}{t} - [(\delta_\zeta^{n-1}+\delta_\zeta^{n-1}\delta_t+\ldots+\delta_t^{n-1})\phi(\zeta)\psi(t)],$$

and therefore more generally, if $f(\;)$ denote any polynomial with *constant* coefficients

$$f(\delta)\int \phi(\zeta)\psi(t)\,\frac{dt}{t} = \int \phi(\zeta)f(\delta_t)\psi(t)\,\frac{dt}{t} - \left[\frac{f(\delta_\zeta)-f(\delta_t)}{\delta_\zeta-\delta_t}\,\phi(\zeta)\psi(t)\right]. \tag{3}$$

* *Acta Soc. Fenn.* 21 (1896), chiefly § 9.

If f_1, f_2, g_1, g_2 are any four such polynomials, we can then write

$$\{f_1(\delta)f_2(\delta)-z^m g_1(\delta)g_2(\delta)\}\int \phi(\zeta)\psi(t)\,\frac{dt}{t}$$

$$=\int \{f_1(\delta_\zeta)f_2(\delta_t)-\zeta^m t^m g_1(\delta_\zeta)g_2(\delta_t)\}\phi(\zeta)\psi(t)\,\frac{dt}{t}-$$

$$-\left[\left\{f_1(\delta_\zeta)\frac{f_2(\delta_\zeta)-f_2(\delta_t)}{\delta_\zeta-\delta_t}-\zeta^m t^m g_1(\delta_\zeta)\frac{g_2(\delta_\zeta)-g_2(\delta_t)}{\delta_\zeta-\delta_t}\right\}\phi(\zeta)\psi(t)\right]. \quad (4)$$

The argument presupposes that the differentiation across the integral-sign is permissible. In the particular applications which follow this can generally be seen to be so without special discussion.

If now ϕ, ψ satisfy the respective differential equations

$$\{f_1(\delta)-z^m g_1(\delta)\}\phi(z)=0, \qquad \{f_2(\delta)-z^m g_2(\delta)\}\psi(z)=0, \quad (5\text{A})$$

the integral on the right of (4) disappears, and we have only to secure the evanescence of the 'terms at the limits' in order to secure that

$$y=\int \phi(\zeta)\psi(t)\,\frac{dt}{t}$$

be a solution of the differential equation

$$\{f_1(\delta)f_2(\delta)-z^m g_1(\delta)g_2(\delta)\}y=0. \quad (5\text{B})$$

If the integral is a contour-integral, the terms at the limits vanish automatically on completion of the contour, provided that the expression in square brackets is one-valued. Our task is then to see that the contour-integral is neither infinite nor zero. If on the other hand the integral is a rectilinear integral, we must choose limits of integration such that the expression in brackets takes the same value (invariably zero, as it happens) at each limit. Here I shall consider only the rectilinear integral.

In the terms at the limits we can write

$$f_1(\delta_\zeta)\frac{f_2(\delta_\zeta)-f_2(\delta_t)}{\delta_\zeta-\delta_t}\phi(\zeta)\psi(t)$$

$$=\frac{f_2(\delta_\zeta)-f_2(\delta_t)}{\delta_\zeta-\delta_t}\zeta^m g_1(\delta_\zeta)\phi(\zeta)\psi(t)$$

$$=\zeta^m g_1(\delta_\zeta)\frac{f_2(\delta_\zeta+m)-f_2(\delta_t)}{\delta_\zeta+m-\delta_t}\phi(\zeta)\psi(t).$$

Thus the expression which must vanish at the limits may be written as

$$\zeta^m g_1(\delta_\zeta)\left\{\frac{f_2(\delta_\zeta+m)-f_2(\delta_t)}{\delta_\zeta+m-\delta_t}-t^m\frac{g_2(\delta_\zeta)-g_2(\delta_t)}{\delta_\zeta-\delta_t}\right\}\phi(\zeta)\psi(t). \quad (6)$$

We may accordingly enunciate *Mellin's principle* as follows:

The differential equation

$$f_1(\delta)f_2(\delta)y = z^m g_1(\delta)g_2(\delta)y$$

is satisfied by the definite integral

$$y = \int \phi(\zeta)\psi(t)\,\frac{dt}{t},$$

if ϕ, ψ are solutions of the respective equations

$$f_1(\delta)\phi = z^m g_1(\delta)\phi, \qquad f_2(\delta)\psi = z^m g_2(\delta)\psi,$$

and if the limits are chosen to make the expression (6) *vanish.*

2. Prolegomena

In illustrating this principle by its applications to Bessel's functions, I confine myself to those integrals which express products of pairs of Bessel's functions. Except in limiting cases I suppose these functions to be of non-integral orders μ, ν.

The differential equations satisfied by products of pairs of Bessel's functions of the same kind are respectively*

$$\{\delta^2-(\mu+\nu)^2\}\{\delta^2-(\mu-\nu)^2\}J_{\pm\mu}J_{\pm\nu} = -4z^2(\delta+1)(\delta+2)J_{\pm\mu}J_{\pm\nu}, \tag{7}$$

$$\{\delta^2-(\mu+\nu)^2\}\{\delta^2-(\mu-\nu)^2\}I_{\pm\mu}I_{\pm\nu} = 4z^2(\delta+1)(\delta+2)I_{\pm\mu}I_{\pm\nu}. \tag{8}$$

If $\mu = \nu$, these equations become respectively

$$\delta^2(\delta^2-4\nu^2)y = -4z^2(\delta+1)(\delta+2)y, \tag{7A}$$

$$\delta^2(\delta^2-4\nu^2)y = 4z^2(\delta+1)(\delta+2)y. \tag{8A}$$

Evidently (7A) has the solutions

$$J_\nu^2, \qquad J_\nu J_{-\nu}, \qquad J_{-\nu}^2,$$

and the 'missing' fourth solution is supplied by

$$J_{-\nu}\frac{\partial J_\nu}{\partial\nu} + J_\nu\frac{\partial J_{-\nu}}{\partial\nu}.$$

If we remove this fourth solution, we get the equation

$$\delta(\delta^2-4\nu^2)y = -4z^2(\delta+1)y, \tag{7B}$$

which has just the three solutions

$$J_\nu^2, \qquad J_\nu J_{-\nu}, \qquad J_{-\nu}^2.$$

By change of sign of z^2 we obtain similar results for (8A) to which corresponds the 'reduced' equation

$$\delta(\delta^2-4\nu^2)y = 4z^2(\delta+1)y. \tag{8B}$$

* Watson, loc. cit., p. 146, 5.4 (3).

The chief analysis of equations (7), (8) turns on the pair of differential equations

$$(\delta^2-\mu^2)E_{\pm\mu} = -z^2(\delta+1)(\delta+2)E_{\pm\mu}, \tag{9}$$

$$(\delta^2-\mu^2)F_{\pm\mu} = z^2(\delta+1)(\delta+2)F_{\pm\mu}, \tag{10}$$

of which the solutions are given by

$$E_\mu(z) = \frac{1}{\sqrt{(1+z^2)}}\left\{\frac{\sqrt{(1+z^2)}+1}{z}\right\}^\mu, \tag{11}$$

$$F_\mu(z) = \frac{1}{\sqrt{(1-z^2)}}\left\{\frac{\sqrt{(1-z^2)}+1}{z}\right\}^\mu. \tag{12}$$

At the origin, we have, if μ is not an integer,

$$E_\mu(z),\ F_\mu(z) = (\tfrac{1}{2}z)^{-\mu}+O(z^{2-\mu}),$$

and at infinity $$E_\mu(z) = z^{-1}+O(z^{-3}).$$

I add for reference that

$$E_\mu(\operatorname{cosech} u) = \tanh u\, e^{\mu u}, \tag{13}$$

$$F_\mu(\operatorname{sech} u) = \cosh u\, e^{\mu u}, \qquad F_\mu(\sec u) = -i\cot u\, e^{i\mu u}. \tag{14}$$

3. Nicholson's integral for $K_\mu K_\nu$

We first analyse (8) into the equations

$$\{\delta^2-(\mu+\nu)^2\}\phi(z) = 4z^2\phi(z), \tag{15}$$

$$\{\delta^2-(\mu-\nu)^2\}\psi(z) = z^2(\delta+1)(\delta+2)\psi(z). \tag{16}$$

Taking $\phi = K_{\mu+\nu}(2z)$, $\psi = F_{\mu-\nu}(z)$, we get the Mellin integral

$$\int K_{\mu+\nu}\left(\frac{2z}{t}\right)F_{\mu-\nu}(t)\,\frac{dt}{t}.$$

The terms at the limits are now

$$4\zeta^2\{(1-t^2)(\delta_\zeta+\delta_t+3)-1\}K_{\mu+\nu}(2\zeta)F_{\mu-\nu}(t).$$

These vanish as $t \to 0+$, if $R(z) > 0$. Writing $t = \operatorname{sech} u$, we therefore have as a solution of (8)

$$\int_{-\infty}^{\infty} K_{\mu+\nu}(2z\cosh u)e^{(\mu-\nu)u}\,du,$$

or more conveniently,

$$\int_0^\infty K_{\mu+\nu}(2z\cosh u)\cosh(\mu-\nu)u\,du.$$

Now $\cosh u \geqslant 1$ and the factor $K_{\mu+\nu}$ therefore assures at $R(z) = +\infty$ a zero of order less than that of any

$$e^{-(2-\epsilon)R(z)} \qquad (\epsilon > 0).$$

The general solution of (8) is a sum of constant multiples of the products

$$K_\mu K_\nu, \qquad K_\mu I_\nu, \qquad I_\mu K_\nu, \qquad I_\mu I_\nu.$$

We obtain a zero of exponential order, only if we take the first of the products in isolation. We therefore write

$$K_\mu(z)K_\nu(z) = C\int_0^\infty K_{\mu+\nu}(2z\cosh u)\cosh(\mu-\nu)u\,du. \tag{17}$$

The value of C is found to be 2 by identifying, on the two sides of (17), the dominant terms at $z=\infty$. We thus reach the final formula:

$$K_\mu(z)K_\nu(z) = 2\int_0^\infty K_{\mu+\nu}(2z\cosh u)\cosh(\mu-\nu)u\,du, \tag{18}$$

where $R(z) > 0$ and μ, ν are unrestricted. This is Nicholson's formula, Watson (loc. cit), p. 440, 13.72 (1).

4. Integrals for $I_\mu I_\nu$ and $J_\mu J_\nu$

Still contemplating the analysis of equation (8) into equations (15), (16), we can utilize $I_{\mu+\nu}(2z)$, the other solution of (15). This leads to the Mellin integral

$$-i\int I_{\mu+\nu}\left(\frac{2z}{t}\right)F_{\mu-\nu}(t)\,\frac{dt}{t}, \text{ say,}$$
$$= \int I_{\mu+\nu}(2z\cos u)e^{i(\mu-\nu)u}\,du,$$

if we write $t = \sec u$. The terms at the limits now become

$$2z^2\{\sin 2u\delta_\zeta + 4\cot u(1+\sin^2 u) - i(\mu-\nu)\cos^2 u\}e^{i(\mu-\nu)u}I_{\mu+\nu}(\zeta),$$

where ζ is $2z\cos u$. These vanish at $u = \pm\frac{1}{2}\pi$ if $R(\mu+\nu+1) > 0$. With this condition we have the new solution of (8):

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} I_{\mu+\nu}(2z\cos u)e^{i(\mu-\nu)u}\,du,$$

or more conveniently

$$\int_0^{\frac{1}{2}\pi} I_{\mu+\nu}(2z\cos u)\cos(\mu-\nu)u\,du.$$

We can expand this integral in powers of z, the indices which occur

being the set $\mu+\nu+2r$, where r is a positive integer or zero. The general solution of (8) is a sum of constant multiples of

$$I_\mu I_\nu, \quad I_\mu I_{-\nu}, \quad I_{-\mu} I_\nu, \quad I_{-\mu} I_{-\nu}.$$

If 2μ, 2ν are not integers, we obtain a sum of powers $z^{\mu+\nu+2r}$ only, only from the first of the products in isolation. We therefore write

$$I_\mu I_\nu = C \int_0^{\frac{1}{2}\pi} I_{\mu+\nu}(2z\cos u)\cos(\mu-\nu)u\,du, \tag{19}$$

extending the results to integer values of 2μ, 2ν by continuity. C is found to be $2/\pi$ by comparing, in (19), the leading terms at $z=0$. We thus get at length

$$\tfrac{1}{2}\pi I_\mu I_\nu = \int_0^{\frac{1}{2}\pi} I_{\mu+\nu}(2z\cos u)\cos(\mu-\nu)u\,du, \tag{20}$$

if $R(\mu+\nu+1)>0$.

If we write $\mu=-\nu$, we deduce

$$\tfrac{1}{2}\pi I_\nu I_{-\nu} = \int_0^{\frac{1}{2}\pi} I_0(2z\cos u)\cos 2\nu u\,du. \tag{21}$$

Again, from (20) we have

$$\tfrac{1}{2}\pi\left(I_\mu \frac{\partial}{\partial\nu} I_\nu + I_\nu \frac{\partial}{\partial\mu} I_\mu\right) = \int_0^{\frac{1}{2}\pi} 2\frac{\partial}{\partial(\mu+\nu)} I_{\mu+\nu}(2z\cos u)\cos(\mu-\nu)u\,du.$$

If again we put $\mu=-\nu$, we deduce

$$\tfrac{1}{4}\pi\left(I_\nu \frac{\partial}{\partial\nu} I_{-\nu} - I_{-\nu}\frac{\partial}{\partial\nu} I_\nu\right) = \int_0^{\frac{1}{2}\pi} K_0(2z\cos u)\cos 2\nu u\,du. \tag{22}$$

We may similarly analyse equation (7) into

$$\{\delta^2-(\mu+\nu)^2\}\phi(z) = -4z^2\phi(z) \tag{23}$$

and equation (16). If we take the solution $J_{\mu+\nu}(2z)$ of (23) and proceed as we have just done with the solution $I_{\mu+\nu}(2z)$ of (15), we arrive at the formula, analogous to (20),

$$\tfrac{1}{2}\pi J_\mu J_\nu = \int_0^{\frac{1}{2}\pi} J_{\mu+\nu}(2z\cos u)\cos(\mu-\nu)u\,du, \tag{24}$$

where $R(\mu+\nu+1)>0$. From this we may similarly deduce

$$\tfrac{1}{2}\pi J_\nu J_{-\nu} = \int_0^{\frac{1}{2}\pi} J_0(2z\cos u)\cos 2\nu u\,du \tag{25}$$

and $$\tfrac{1}{4}\left(J_{-\nu}\frac{\partial}{\partial\nu}J_{\nu}-J_{\nu}\frac{\partial}{\partial\nu}J_{-\nu}\right)=\int\limits_{0}^{\frac{1}{2}\pi}Y_0(2z\cos u)\cos 2\nu u\,du. \tag{26}$$

Of the foregoing formulae, (20) and (24) are those given by Watson (loc. cit.), p. 441, 13.72, (2), and p. 150, 5.43, (1) respectively.

5. The Nicholson-Watson integrals

Thus far the analysis of the equations (7), (8) has been based on equation (10) satisfied by the algebraic function which I have called F_{μ}. I now go on to an analysis based on equation (9) satisfied by the analogous algebraic function E_{μ}.

Consider equation (7), and analyse it into the two equations

$$\{\delta^2-(\mu-\nu)^2\}\phi(z)=4z^2\phi(z),$$
$$\{\delta^2-(\mu+\nu)^2\}\psi(z)=-z^2(\delta+1)(\delta+2)\psi(z).$$

Taking $\phi=K_{\mu-\nu}(2z)$, $\psi=E_{\mu+\nu}(z)$ we get the Mellin integral

$$\int K_{\mu-\nu}\left(\frac{2z}{t}\right)E_{\mu+\nu}(t)\,\frac{dt}{t}.$$

The terms at the limits are

$$4\zeta^2\{(1+t^2)(\delta_{\zeta}+\delta_t+3)-1\}K_{\mu-\nu}(2\zeta)E_{\mu+\nu}(t).$$

These vanish as $t\to+0$, if $R(z)>0$, and as $t\to+\infty$, if $|R(\mu-\nu)|<1$. We have therefore the solution of (7)

$$\int\limits_{0}^{\infty}K_{\mu-\nu}\left(\frac{2z}{t}\right)E_{\mu+\nu}(t)\,\frac{dt}{t}, \tag{27}$$

if $R(z)>0$, $|R(\mu-\nu)|<1$.

If we write $t=z\tau$ in (27) and remember that every $E(z)$ is of order z^{-1} at $z=\infty$, we see that the integral (27) is itself of the order z^{-1}. More conveniently we write $t=\operatorname{cosech}u$ and replace (27) by the integral

$$\int\limits_{0}^{\infty}K_{\mu-\nu}(2z\sinh u)e^{-(\mu+\nu)u}\,du. \tag{28}$$

Now the general solution of (7) is a sum of constant multiples of the products

$$J_{\mu}J_{\nu},\qquad J_{\mu}Y_{\nu},\qquad Y_{\mu}J_{\nu},\qquad Y_{\mu}Y_{\nu}.$$

At infinity the dominant terms of J_{ν}, Y_{ν} are

$$J_{\nu}\sim\sqrt{\left(\frac{2}{\pi z}\right)}\cos(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi),\qquad Y_{\nu}\sim\sqrt{\left(\frac{2}{\pi z}\right)}\sin(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi).$$

Thus only in $$J_\mu Y_\nu - J_\nu Y_\mu, \qquad J_\mu J_\nu + Y_\mu Y_\nu \tag{29}$$
and their linear multiples are the dominant terms free of the oscillatory factors $\cos z$, $\sin z$. They have each the same order as (28), namely, z^{-1}, and therefore (28) is a sum of constant multiples of the two expressions (29). We can distinguish between the two expressions, and determine the appropriate constant multipliers by considering the second terms in the asymptotic expansions. We finally get that

$$\int_0^\infty K_{\mu-\nu}(2z\sinh u)e^{-(\mu+\nu)u}\,du = \tfrac{1}{4}\pi^2 \operatorname{cosec}(\mu-\nu)\pi(J_\mu Y_\nu - J_\nu Y_\mu), \tag{30}$$

if $R(z) > 0$, $|R(\mu-\nu)| < 1$.

By changing the signs of μ, ν and expressing $J_{-\mu}$, $Y_{-\mu}$, etc., in terms of J_μ, Y_μ, etc., we deduce that

$$\int_0^\infty K_{\mu-\nu}(2z\sinh u)\cosh(\mu+\nu)u\,du$$
$$= \tfrac{1}{8}\pi^2\{J_\mu J_\nu + Y_\mu Y_\nu + \tan\tfrac{1}{2}(\mu-\nu)\pi(J_\mu Y_\nu - J_\nu Y_\mu)\}, \tag{31}$$

subject to the same conditions on z, $\mu-\nu$. When $\mu = \nu$, these formulae become

$$\int_0^\infty K_0(2z\sinh u)e^{-2\nu u}\,du = \tfrac{1}{4}\pi\left\{Y_\nu \frac{\partial}{\partial\nu}(J_\nu) - J_\nu \frac{\partial}{\partial\nu}(Y_\nu)\right\}, \tag{32}$$

$$\int_0^\infty K_0(2z\sinh u)\cosh 2\nu u\,du = \tfrac{1}{8}\pi^2(J_\nu^2 + Y_\nu^2), \tag{33}$$

provided that $R(z) > 0$.

Of these formulae (30), (31) are effectively those given by Watson (loc. cit.) as p. 445, 13.73 (4), (5), while (32), (33) are, of course, the formulae p. 444, 13.73 (1), (2).

6. An integral involving $W_{k,m}$

There are two other modes of splitting up equations (7), (8) which lead to results apparently new but not, I think, otherwise of interest. In the first place we analyse (8) into

$$\{\delta^2 - (\mu-\nu)^2\}\phi(z) = -4z^2(\delta+2)\phi(z), \tag{34}$$
$$\{\delta^2 - (\mu+\nu)^2\}\psi(z) = -z^2(\delta+1)\psi(z). \tag{35}$$

These are equations of confluent-hypergeometric type. In point of fact (34) has the solutions $$z^{-1}e^{-\frac{1}{2}z^2}W_{\frac{1}{2},\,\pm\frac{1}{2}(\mu-\nu)}(\tfrac{1}{2}z^2) \tag{36}$$

in Whittaker's notation,* while solutions of (35) are

$$e^{-z^2}I_{\frac{1}{2}(\mu+\nu)}(z^2), \qquad e^{-z^2}K_{\frac{1}{2}(\mu+\nu)}(z^2). \tag{37}$$

I am doubtful of the interest of these functions and I accordingly give the analysis only for the case $\mu=\nu$. We then replace equation (8) by the third-order equation (8 B), which we analyse into

$$\delta\phi(z) = -z^2\phi(z),$$

$$(\delta^2-4\nu^2)\psi(z) = -4z^2(\delta+1)\psi(z).$$

Taking $\qquad \phi(z) = e^{-\frac{1}{2}z^2}, \qquad \psi(z) = e^{-z^2}K_\nu(z^2),$

we consider the integral

$$\int e^{-\frac{1}{2}z^2t^{-2}-t^2}K_\nu(t^2)\,\frac{dt}{t}.$$

The terms at the limits are now

$$\zeta^2(\delta_\zeta+\delta_t+4t^2+2)e^{-\frac{1}{2}\zeta^2-t^2}K_\nu(t^2),$$

which, in virtue of the exponential factors, vanish both at $\zeta=0$ and at $t=0$, if $R(z^2)>0$. We therefore have, as a solution of (8 B), the integral

$$\int_0^\infty e^{-\frac{1}{2}z^2t^{-2}-t^2}K_\nu(t^2)\,\frac{dt}{t},$$

or, more conveniently, by writing $t^2=\frac{1}{2}zt'$ and then dropping accents, the integral

$$\int_0^\infty e^{-(\frac{1}{2}t+t^{-1})z}K_\nu(\tfrac{1}{2}zt)\,\frac{dt}{t} \qquad [R(z)>0]. \tag{38}$$

At $R(z)=+\infty$ the dominant term is

$$\sqrt{\left(\frac{\pi}{z}\right)}\int_0^\infty e^{-(t+t^{-1})z}t^{-\frac{3}{2}}\,dt.$$

If we fold this integral over at $t=1$ and write $u=t^{\frac{1}{2}}-t^{-\frac{1}{2}}$, it becomes

$$2\sqrt{\left(\frac{\pi}{z}\right)}\int_0^\infty e^{-(u^2+2)z}\,du,$$

which reduces without difficulty to

$$\pi z^{-1}e^{-2z}.$$

Now the general solution of (8 B) is a sum of constant multiples of

* Whittaker and Watson, *Modern Analysis*, p. 336, 16.1.

I_ν^2, $I_\nu K_\nu$, K_ν^2. Clearly K_ν^2 itself is the only combination which has the appropriate form at infinity, and comparing coefficients we write

$$2K_\nu^2(z) = \int\limits_0^\infty e^{-(\frac{1}{2}t+t^{-1})z} K_\nu(\tfrac{1}{2}zt)\,\frac{dt}{t}, \tag{39}*$$

if $R(z) > 0$.

The more general result, deducible from equation (8) when $\mu \neq \nu$, appears to be

$$2K_\mu(z)K_\nu(z) = \int\limits_0^\infty e^{-\frac{1}{2}(t+t^{-1})z} W_{\frac{1}{4},\frac{1}{2}(\mu-\nu)}(zt^{-1}) K_{\frac{1}{2}(\mu+\nu)}(\tfrac{1}{2}zt)\,\frac{dt}{\sqrt{(zt)}}, \tag{40}$$

where $R(z) > 0$.

The corresponding analysis of (7) or (7B) does not seem to yield a rectilinear integral.

7. An integral involving a hypergeometric function

My last analysis is that of equation (7) into the equations

$$(\delta-\mu+\nu)(\delta-\mu-\nu)\phi(z) = -4z^2\phi(z), \tag{41}$$

$$(\delta+\mu-\nu)(\delta+\mu+\nu)\psi(z) = z^2(\delta+1)(\delta+2)\psi(z). \tag{42}$$

The first of these has the solution

$$\phi(z) = z^\mu J_\nu(2z); \tag{43}$$

the second has the solution

$$\psi(z) = z^{-(\mu+\nu)}F(\tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu, 1-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu; 1-\nu; z^2). \tag{44}$$

Here again I give the analysis only for the simpler case in which $\mu = \nu$. I then analyse (7B) into the equations

$$\delta(\delta-2\nu)\phi(z) = -4z^2\phi(z),$$

$$(\delta+2\nu)\psi(z) = z^2(\delta+1)\psi(z),$$

and take $\phi(z) = z^\nu J_\nu(2z)$, $\psi(z) = z^{-2\nu}(1-z^2)^{\nu-\frac{1}{2}}$,

so that we have the Mellin integral

$$\int \left(\frac{z}{t}\right)^\nu J_\nu\left(\frac{2z}{t}\right) t^{-2\nu}(t^2-1)^{\nu-\frac{1}{2}}\,\frac{dt}{t}.$$

Write $t = \operatorname{cosec} u$, and consider instead the integral

$$z^\nu \int J_\nu(2z\sin u)\sin^\nu u\cos^{2\nu} u\,du.$$

The terms at the limits are now simply

$$\zeta^2(t^2-1)\zeta^\nu J_\nu(2\zeta)t^{-2\nu}(1-t^2)^{\nu-\frac{1}{2}}$$

$$\propto z^{\nu+2}\sin^{\nu+1}u\cos^{2\nu+1}u J_\nu(2z\sin u).$$

* A particular case of Watson, loc. cit., p. 439, 13.71, with $Z = z$, $v = 2/t$.

Near $\sin u = 0$, this is comparable with $\sin^{2\nu+1} u$, and therefore the terms at the limits vanish when these limits are taken to be 0, $\frac{1}{2}\pi$, provided that $R(\nu) > -\frac{1}{2}$. We thus have as a solution of (7B) the integral

$$z^{\nu} \int_0^{\frac{1}{2}\pi} J_{\nu}(2z \sin u) \sin^{\nu} u \cos^{2\nu} u \, du, \tag{45}$$

if $R(\nu) > -\frac{1}{2}$.

If we expand J_{ν} under the sign of integration, we express (45) as a series of powers $z^{2\nu+2r}$, where r is a positive integer. The solutions of (7B) are J_{ν}^2, $J_{\nu} J_{-\nu}$, $J_{-\nu}^2$, and of these J_{ν}^2 alone, if ν is not an integer, is composed of just the right powers of z. We therefore identify (45) with a numerical multiple of J_{ν}^2 and compare coefficients. The leading coefficient in the integral is

$$\{\Gamma(\nu+1)\}^{-1} \int_0^{\frac{1}{2}\pi} \sin^{2\nu} u \cos^{2\nu} u \, du = \frac{\{\Gamma(\nu+\frac{1}{2})\}^2}{2\Gamma(\nu+1)\Gamma(2\nu+1)},$$

while that of J_{ν}^2 is $\qquad 2^{-2\nu}\{\Gamma(\nu+1)\}^{-2}$.

We therefore complete the identification in the form

$$J_{\nu}^2(z) = \frac{2z^{\nu}}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \int_0^{\frac{1}{2}\pi} J_{\nu}(2z \sin u) \sin^{\nu} u \cos^{2\nu} u \, du, \tag{46}$$

where $R(\nu) > \frac{1}{2}$.

If we similarly analyse (8B) into

$$\delta(\delta - 2\nu)\phi(z) = 4z^2 \phi(z),$$
$$(\delta + 2\nu)\psi(z) = z^2(\delta + 1)\psi(z),$$

we merely write I for J everywhere in the preceding argument and so finally achieve the result

$$I_{\nu}^2(z) = \frac{2z^{\nu}}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \int_0^{\frac{1}{2}\pi} I_{\nu}(2z \sin u) \sin^{\nu} u \cos^{2\nu} u \, du, \tag{47}$$

where $R(\nu) > \frac{1}{2}$.

These formulae, (46), (47), can be verified by expansion and direct integration.

I have to thank Professor A. L. Dixon for references and suggestions.

THE VALIDITY OF LAGRANGE'S EXPANSION

By T. W. CHAUNDY

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In the theory of the real variable Lagrange's expansion is usually given in the approximative form that (under certain conditions)

$$f(y)=f(a)+\sum_{r=1}^{n}\frac{x^r}{r!}\frac{d^{r-1}}{da^{r-1}}\{[\phi(a)]^r f'(a)\}+o(x^n), \tag{1}$$

where y is defined implicitly by the relation

$$y=a+x\phi(y), \tag{2}$$

but as a rule nothing is said about the conditions, if any, in which the exact equation

$$f(y)=f(a)+\sum_{r=1}^{n}\frac{x^r}{r!}\frac{d^{r-1}}{da^{r-1}}\{[\phi(a)]^r f'(a)\} \tag{3}$$

is true.

The proof of (1) consists in showing that the series on the right is the Maclaurin-Taylor expansion of $f(y)$, regarded, in virtue of (2), as a function of x. To establish (3) we have therefore to prove that $f(y)$ is exactly represented by this particular Taylor series. Conditions under which a function of a real variable is exactly the sum of one of its Taylor's series have been published by Pringsheim* and by Young.† The form of condition which I shall use here is one which I have given elsewhere.‡

The Taylor series

$$\sum_{0}^{\infty}F^{(n)}(a)\frac{(x-a)^n}{n!},$$

if it converges, converges to $F(x)$, provided that x, a both lie in an interval in which $F(x)$ has derivatives $F^{(n)}(x)$ of every order, and

$$|F^{(n)}(x)/n!|^{1/n},$$

regarded as a function of x and positive integral n, is bounded.

* *Math. Annalen*, vol. 44, p. 57.

† *Quart. J. of Math.*, vol. 40, p. 157 : 'The fundamental theorems of the differential calculus' (1910), p. 57.

‡ *Messenger of Math.*, vol. 45 (1915), p. 115. S. Bernstein (*Math. Annalen*, vol. 75, p. 449) quotes this particular form as from Pringsheim '*Math. Annalen*, vol. 45' where I cannot trace it. Pringsheim (loc. cit.) gives a variety of equivalent forms, but, if I follow him, does not explicitly mention this particular form.

It is convenient to borrow from the theory of the complex variable and say that $F(x)$ is 'analytic' in an interval in which every $F^{(n)}(x)$ exists and $|F^{(n)}(x)/n!|^{1/n}$ is bounded. We are then reminded that the theory of the real analytic function is implicit in that of the complex analytic function and novelties are not to be looked for. There are reasons, however, for discussing the real analytic function independently of the more inclusive theory.

For our present purpose, namely to prove the validity of the Maclaurin-Taylor expansion of $F(x) \equiv f(y)$, we have to prove $F(x)$ analytic in the neighbourhood of $x = 0$. We begin by proving inductively that the nth derivative $F^{(n)}(x)$ is given by the formula

$$F^{(n)}(x) = \left\{\frac{1}{1-x\phi'(y)}\frac{\partial}{\partial y}\right\}^{n-1}\frac{f'(y)[\phi(y)]^n}{1-x\phi'(y)}, \tag{4}$$

presupposing the necessary derivatives on the right.

Now from (2) we have

$$\frac{dy}{dx} = \frac{\phi(y)}{1-x\phi'(y)} \quad \text{and so} \quad \frac{d}{dx} = \frac{\partial}{\partial x} + \frac{\phi(y)}{1-x\phi'(y)}\frac{\partial}{\partial y}.$$

Writing $P|Q$ for the alternant $PQ-QP$ of operators P, Q, we have

$$\frac{\partial}{\partial x}\left|\frac{1}{1-x\phi'}\frac{\partial}{\partial y} = \frac{\phi'}{(1-x\phi')^2}\frac{\partial}{\partial y},\right.$$

$$\frac{\phi}{1-x\phi'}\frac{\partial}{\partial y}\left|\frac{1}{1-x\phi'}\frac{\partial}{\partial y} = \frac{-\phi'}{(1-x\phi')^2}\frac{\partial}{\partial y},\right.$$

and so, by addition,
$$\frac{d}{dx}\left|\frac{1}{1-x\phi'}\frac{\partial}{\partial y} = 0,\right.$$

i.e. these two operators are commutative. Thus for operation on any differentiable function $\psi(y)$ we get

$$\frac{d}{dx}\left\{\frac{1}{1-x\phi'}\frac{\partial}{\partial y}\right\}^n\frac{\psi'(y)}{1-x\phi'} = \left\{\frac{1}{1-x\phi'}\frac{\partial}{\partial y}\right\}^{n+1}\frac{d}{dx}\psi(y)$$
$$= \left\{\frac{1}{1-x\phi'}\frac{\partial}{\partial y}\right\}^{n+1}\frac{\phi\psi'(y)}{1-x\phi'}.$$

Hence, if we here write $\psi'(y) \equiv f'(y)[\phi(y)]^n$, which is legitimate, since $f'\phi^n$, being differentiable, is also integrable, we have proved that

$$\frac{d}{dx}Y_n = Y_{n+1},$$

where Y_n stands for the expression on the right of (4). Again

$$Y_1 = \frac{\phi(y)f'(y)}{1-x\phi'(y)} = \frac{d}{dx}f(y),$$

which is correct. The induction is therefore complete and the formula (4) proved.

We use (4) to prove the essential theorem:

(1) *If for some* x, y

$$|f^{(n)}(y)|,\ |\phi^{(n)}(y)| < n!AB^n \qquad (\textit{every } n \geqslant 0),$$

and $$|1 - x\phi'(y)|^{-1} < C,$$

then $$|F^{(n)}(x)| < n!A_1 B_1^n \qquad (\textit{every } n \geqslant 0),$$

where A, B, C, A_1, B_1 *denote positive constants independent of* x, y, n.

The proof of (1) is based on the two lemmas:

(2) *If* $$|f^{(n)}(y)|,\ |\phi^{(n)}(y)| < n!AB^n \qquad (\textit{every } n \geqslant 0),$$

and p *is integral,*

then $$|D^n(f'\phi^p)| < n!A_1^{p+1}B_1^{n+1} \qquad (\textit{every } n, p \geqslant 0),$$

where D *denotes differentiation in* y, *and* A_1, B_1 *are positive constants independent of* x, n, p;

and

(3) *If* $$|f^{(n)}(y)| < n!A_0 B^n, \qquad |z^{(n)}(y)| < n!AB^n \quad (\textit{every } n \geqslant 0),$$

and $$\left|\frac{dz}{dy}\right|^{-1} < C,$$

then $$\left|\frac{d^n f}{dz^n}\right| < n!A_0 A_1 B_1^n \qquad (\textit{every } n \geqslant 0),$$

where A_1, B_1 *are independent of* x, n, A_0.

To prove (2) we have by Leibniz's formula

$$D^n(f'\phi) = n! \sum_{s=0}^{n} \frac{f^{(s+1)}\phi^{(n-s)}}{s!(n-s)!},$$

i.e. $$|D^n(f'\phi)| < n!A^2B^{n+1} \sum_{s=0}^{n} (s+1) = \frac{(n+2)!}{2!} A^2 B^{n+1}.$$

Again $$D^n(f'\phi^2) = n! \sum_{s=0}^{n} \frac{(f'\phi)^{(s)}\phi^{(n-s)}}{s!(n-s)!},$$

i.e. $$|D^n(f'\phi^2)| < n!A^3B^{n+1} \sum_{s=0}^{n} \frac{(s+2)!}{2!s!} = \frac{(n+3)!}{3!} A^3 B^{n+1}.$$

Proceeding inductively in this way we get at length for any positive integer p

$$|D^n(f'\phi^p)| < \frac{(n+p+1)!}{(p+1)!} A^{p+1} B^{n+1}.$$

Now $\dfrac{(n+p+1)!}{n!(p+1)!}$ is a coefficient in the expansion of $(1+x)^{n+p+1}$ and is therefore certainly less than 2^{n+p+1}, the sum of the coefficients in the expansion. We thus have that

$$|D^n(f'\phi^p)| < n!(2A)^{p+1}(2B)^{n+1},$$

which proves the lemma (**2**).

To prove (**3**) we write

$$\frac{d^n f}{dz^n} = \sum_{r=0}^{n-1}\sum_{s=0}^{r}(-)^s Z_{n,r,s}\left(\frac{dz}{dy}\right)^{-(n+s)}\frac{d^{n-r}f}{dy^{n-r}}, \tag{5}$$

where $Z_{n,r,s}$ is a function of arguments $\dfrac{d^2z}{dy^2}, \dfrac{d^3z}{dy^3},\ldots$ only. Differentiation and comparison of coefficients give the recurrence-formula

$$Z_{n+1,r,s} = Z_{n,r,s} + D_y Z_{n,r-1,s} + (n+s-1)(D_y^2 z)Z_{n,r-1,s-1} \tag{6}$$

which, used inductively, shows $Z_{n,r,s}$ to be a polynomial with positive numerical coefficients.

Let us write $$u_m \equiv (D_y^m z)/m!, \tag{7}$$

so that $Z_{n,r,s}$ becomes a polynomial in u_m $(m = 2, 3,\ldots)$ with positive coefficients. Its degree and weight in u_m being its orders in z and y^{-1} respectively, we see from (5) that $Z_{n,r,s}$ is homogeneous of degree s and isobaric of weight $r+s$.

Since, from (7), $$D_y u_m = (m+1)u_{m+1},$$

the sum of the coefficients in $D_y \Pi u_r^{a_r}$ is

$$\sum (r+1)a_r = i+w,$$

where i, w are the degree and weight of $\Pi u_r^{a_r}$. It follows that, if c is the sum of the coefficients in some $f(u_2, u_3,\ldots)$ which is homogeneous and isobaric of degree i and weight w, then $(i+w)c$ is the sum of the coefficients in $D_y f$.

Hence, if $c_{n,r,s}$ is the sum of the coefficients in $Z_{n,r,s}$, the sum in $D_y Z_{n,r,s}$ is $(r+2s)c_{n,r,s}$. From (6) we therefore derive the recurrence-formula

$$c_{n+1,r,s} = c_{n,r,s} + (r+2s-1)c_{n,r-1,s} + 2(n+s-1)c_{n,r-1,s-1},$$

the coefficients being additive, since they are all positive. This recurrence-formula, together with the boundary values

$$c_{n,r,s} = 0, \text{ if } r > n-1 \text{ or } s > r, \quad c_{n,0,0} = 1,$$

defines inductively a unique $c_{n,r,s}$. By scientific, rather than mathematical, induction I discover the formula

$$c_{n,r,s}=\frac{(n+s-1)!(r-1)!,}{(n-r-1)!(r-s)!s!(s-1)!},$$

which can be verified by sheer substitution in the defining equations.

Since $|u_m|<AB^m$ by the hypothesis of (**3**), we have

$$|Z_{n,r,s}|<c_{n,r,s}A^sB^{r+s},$$

while from (5) we have

$$\left|\frac{d^nf}{dz^n}\right|<\left|\frac{dz}{dy}\right|^{-(2n-1)}\sum_{r=0}^{n-1}\left|\frac{d^{n-r}f}{dy^{n-r}}\right|\cdot\left|\frac{dz}{dy}\right|^{n-r-1}\sum_{s=0}^{r}\left|Z_{n,r,s}\right|\cdot\left|\frac{dz}{dy}\right|^{r-s}.$$

Since also by the hypothesis of (**3**)

$$\left|\frac{d^mf}{dy^m}\right|<m!A_0B^m,\qquad \left|\frac{dz}{dy}\right|^{-1}<C,\qquad \left|\frac{dz}{dy}\right|<AB,$$

we get

$$\left|\frac{d^nf}{dz^n}\right|<C^{2n-1}\sum_{r=0}^{n-1}(n-r)!A_0B^{n-r}(AB)^{n-r-1}\sum_{s=0}^{r}c_{n,r,s}A^sB^{r+s}(AB)^{r-s},$$

i.e. $$<A_0A^{n-1}(BC)^{2n-1}\sum_{r=0}^{n-1}(n-r)!\sum_{s=0}^{r}c_{n,r,s}.$$

Now

$$\sum_{r=0}^{n-1}(n-r)!\sum_{s=0}^{r}c_{n,r,s}=\sum_{r=0}^{n-1}(n-1)!(n-r)\sum_{s=0}^{r}\frac{(n+s-1)!(r-1)!}{(n-1)!s!(r-s)!(s-1)!}.$$

This is the coefficient of t^{n-1} in

$$\sum_{r=0}^{n-1}(n-1)!(n-r)\sum_{s=0}^{r}\frac{(r-1)!}{(r-s)!(s-1)!}(1+t)^{n+s-1}$$
$$=\sum_{r=0}^{n-1}(n-1)!(n-r)(1+t)^n(2+t)^{r-1}.$$

This expression is less, term by term, than

$$n!(1+t)^n\sum_{r=0}^{n-1}(2+t)^{r-1}=n!(1+t)^{n-1}\{(2+t)^{n-1}-1\}.$$

The coefficient of t^{n-1} is certainly less than the sum of the coefficients, and so less than $n!2^{n-1}3^{n-1}$. Hence finally

$$\left|\frac{d^nf}{dz^n}\right|<n!A_0\frac{(6AB^2C^2)^n}{6ABC},$$

which proves the lemma (**3**).

To prove (**1**), write as before

$$\psi'(y)\equiv f'(y)[\phi(y)]^p$$

for positive integral p. Then by (**2**)

$$|\psi^{(n)}(y)| < (n-1)!A_1^{p+1}B_1^n.$$

Write $$z(y) \equiv y - x\phi(y),$$

where x is now to be regarded as a constant. Then

$$|z^{(n)}(y)| = |x|\,.\,|\phi^{(n)}(y)| < n!A_2B_2^n,$$

and therefore by (**3**)

$$\left|\left\{\frac{1}{1-x\phi'(y)}\frac{\partial}{\partial y}\right\}^n \psi(y)\right| < n!A_1^{p+1}A_3B_3^n.$$

Now take $p = n$ and we have

$$|F^{(n)}(x)| = \left|\left\{\frac{1}{1-x\phi'(y)}\frac{\partial}{\partial y}\right\}^n \psi(y)\right| < n!A_4B_4^n,$$

which proves (**1**).

It remains to translate (**1**) into its permanent form:

(**4**) *If the functions $f(y)$, $\phi(y)$ are analytic near $y = a$, and if $y(x)$ is defined implicitly by the relation*

$$y = a + x\phi(y),$$

where, if $y(x)$ is many-valued, we mean that branch which has the value a at $x = 0$, then y, or more generally $f(y)$, is an analytic function of x near $x = 0$ and is exactly equal to the sum-to-infinity of its Lagrange series.

Since $\phi'(y)$ exists and is continuous near $y = a$, and $1 - x\phi'(y)$ does not vanish at $x = 0$, then, by the theory of implicit functions,* the relation

$$y = a + x\phi(y)$$

defines near $x = 0$ a one-valued, continuous function $y(x)$ such that $y(a) = 0$. If x is sufficiently small, we can therefore secure both that

$$1 - x\phi'(y) > \text{some } 1 - C > 0,$$

and that $y(x)$ lies in a prescribed neighbourhood of $y = a$. The conditions of (**1**) are then satisfied and accordingly $F(x)$ is analytic in this neighbourhood of $x = 0$ and is represented accurately by its Maclaurin-Taylor expansion, which is Lagrange's series for $f(y)$.

The foregoing analysis can be extended without difficulty to show that certain mixed functions of x, y such, for instance, as

$$[1 - x\phi'(y)]^{-1}$$

are also analytic functions of x near $x = 0$ and are represented by the appropriate modification of Lagrange's series.

* Cf. Goursat, *Cours d'analyse* (1910), vol. i, § 34, p. 81.